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THE

FIRST AND SECOND BOOKS

OF

EUCLID

EXPLAINED TO BEGINNERS.

BY

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PREFACE.

ATTENTION has been strongly directed of late to the mode in which geometry is usually taught in this country, and several vigorous attempts have been made to supersede Euclid as a text-book. Euclid's method has been objected to, as being needlessly rigid, and consequently needlessly difficult for beginners, some of the most serious difficulties unfortunately being encountered in the earliest propositions. It has been supposed that by extending the field embraced in the postulates, allowing the use of *hypothetical* constructions, and adding two or three axioms, the truth of which no one will dispute, many, if not all of these difficulties may be avoided. It may be conceded that Euclid's method is difficult; but those who object to this should remember that the object of the study of geometry is not merely to obtain a knowledge of the relations and properties of geometrical figures, but to train the logical and perceptive faculties by a course of the most rigid and exact reasoning. The elimination of all the difficulties would deprive the study of nearly all its utility. It takes a beginner a long time to master the first six or seven propositions in Euclid, but it is quite a mistake to suppose that his real progress in geometry, as regards its higher and more important uses, is to be measured by the number of propositions thus learnt. The first six propositions in

Euclid contain as much real geometrical *study* as three or four times that number in some other systems that might be mentioned.

It must also be remembered that Euclid's *method* is not identical with Euclid's *text*. The former may be retained in all its essential features, while the latter may have its phraseology or arrangement modified, so as to make the course of the reasoning more evident to a beginner. This is what has been attempted in the present work. The author has endeavoured to enable the learner to gain an accurate idea of the chain of reasoning in each proposition, to get him to discriminate clearly between the data and the inferences, to point out the logical pitfalls that he must guard against, and, generally, to present the propositions in the shape which would probably be given to them by an intelligent teacher, who was endeavouring to make a beginner understand the text of Euclid. It requires some experience in teaching and examining boys to become aware in how many cases learners who have "gone through Euclid," and can write out the propositions with perfect exactness, are absolutely destitute of any intelligent comprehension of the reasoning. The result achieved is a mere effort of memory. Over and over again the author has met with those who could write out or go through the fifth proposition of the first book, as it stands in the text of Euclid, without a mistake, and yet had not even the slightest idea that the enunciation contained a statement of something that was known to begin with, and something else which had yet to be deduced from the data.

By way of introduction to each proposition, the various axioms, postulates, and propositions which are assumed in the course of the construction or demonstration are stated at length. Many may think this superfluous; but it will usually be found that when boys are set to prepare propositions from the text of the ordinary editions of Euclid, not more than one in twenty gives himself the trouble to recollect, or refer to the previous propositions which are employed in any given demonstration. By setting forth explicitly and at length all that is assumed as known in the course of a demonstration, the mutual dependence of the propositions is far more clearly perceived, and each proposition becomes an independent whole, admitting of being comprehended and mastered without a knowledge of the mode in which the subordinate propositions are to be established.

In teaching geometry I attach very great importance to the repetition of the propositions with figures to different shapes and in different positions. Another point which should be rigidly insisted on is, that in all cases in which *construction* is required, the illustrative diagram should be drawn step by step according to the directions given. In the definition of angular magnitude some eminent teachers avail themselves of the conception of the revolution of a straight line from a given position round a fixed point in the line itself. This furnishes the beginner with an admirable *illustration*, but I conceive it is a mistake to embody it in a *definition*, since the conception of an angle may be formed independently of all idea of motion or revolution; just as it would be unnecessary (not to say incorrect) to define a line as produced by the motion of a point. In fact it is a mistake to say that the motion of a point produces a line. All that it *produces* is a change in the position of the point. The *line* along which the point has moved is a separate creation of the mind. In like manner the revolution of a line does not *produce* angular magnitude; it produces only a change in the position of the line.

Angular magnitude is a separate product of thought, for which a certain relative *position* (not *motion*) of two lines is essential.

Whatever may be the difficulties or defects of Euclid's "Elements," it is tolerably certain that it will be a long time before it is superseded as a text-book. Meanwhile it is of importance that those who have to use the book should understand it as thoroughly as possible. It is hoped that the present treatise may help many towards the attainment of that object.

The greater part of this work is a second edition of "The First Book of Euclid explained to Beginners;" to this has been added an investigation of the Second Book, conducted on similar principles.

C. P. MASON.

THE

FIRST BOOK OF EUCLID

EXPLAINED TO BEGINNERS.

THE First Book of Euclid treats of the properties of figures made up of points, straight lines, and circles, and drawn in one and the same plane.

DEFINITIONS.

Def. I. A point is something which is indivisible, that is, which has no magnitude.

Def. II. A line is that which has length without breadth.

Def. V. A superficies, or surface, is that which has length and breadth, but not thickness.

The use of these definitions is to caution us that in geometry we are not reasoning about such figures as we can draw with a pen or pencil upon a sheet of paper. Figures of this kind are used in the study of geometry, but only for the purpose of helping the memory and imagination. The propositions of geometry can only be *absolutely* true of figures that are *absolutely* perfect. But absolutely perfect figures can only be imagined. It is impossible even to get a *continuous surface*. The only physical surfaces that we can deal with are made up of atoms lying *very close* to each other, but *not absolutely in contact*. The finest stroke that is visible must have a certain amount of breadth, or it could not reflect light or be visible at all; and moreover, no line that we can draw is without break, for it must be drawn upon some surface, and no visible surface is continuous. The smallest dot that can be made is a spot of some size. But these rude material surfaces, lines, and points *suggest* to our minds the *perfect* surfaces, lines, and points which we deal with in geometry. The definitions of these given by Euclid are entirely *negative* in their character, and warn us that we must correct our first rough notion of a point till, by an effort of the mind, we have got rid of the idea of *size* altogether, and in like manner correct our first rough notion of a line till we have got rid of the idea of *breadth*. The use of the figures drawn in geometry is to aid us in *imagining* those *perfect* figures about which we really reason.

As the surfaces and lines treated of in geometry exist only in our conceptions, and are not actual material objects, we can speak of surfaces and lines of *infinite* extent, that is, having no limits or termination.

Def. III. The extremities of a finite or bounded line are points.

Def. VI. The extremities of a finite or bounded surface are lines.

Def. IV. A right (or straight) line is one which lies evenly between its extremities.

It is impossible to define the term *right* or *straight* without using some equivalent word which needs definition quite as much as *right* or *straight* itself. Our notion of *straight* is one of those primary conceptions of form which cannot be analysed into simpler elements. Euclid's definition is, moreover, faulty, inasmuch as it would not be applicable to a straight line of infinite length, which has no extremities.

Def. VII. A plane surface is one which lies evenly between its extremities.

This definition errs in the same manner as the last. It is better to define a plane surface as a surface such, that if any two points be taken upon it, the straight line drawn from the one point to the other lies entirely on the surface.

Def. A. Any combination of lines drawn upon a plane is called a plane figure.

Def. XIV. A plane bounded figure is a plane surface bounded by a line or lines.

Def. XX. A plane rectilineal figure is a figure composed of straight lines, drawn upon the same plane.

Def. B. The distance between two points is the length of a straight line drawn from one to the other. A straight line drawn from one point to another is said to *join* those two points.

Def. C. The area of a bounded figure is the extent of surface enclosed within its boundary.

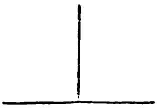
Def. VIII. A rectilinear angle is the opening formed between two straight lines which meet. It is also defined sometimes to be the inclination of one straight line to another.

Beginners often find difficulty in properly understanding the nature of angular magnitude. The size of an angle has nothing to do with the length of the lines by the meeting of which it is formed, nor has it anything to do with the quantity of space which is (at least partially) enclosed by these lines. Perhaps the best way of explaining the matter to a beginner is the following:—Take a pair of compasses; when the legs are shut close together, no angle is formed between them; begin to open the compasses, and one leg forms an angle at the joint with the other leg. The more the compasses are opened the larger this angle becomes —the greater is the slope or inclination which one leg has to the other. It is evident 'at once that this increasing angle does not depend for size upon the length of the lines that form it, for the legs of the compasses remain of the same length, however widely they may be opened; and any angle formed by them would remain just the same in size if half the legs were broken off.

Def. D. Two angles are equal when they admit of being placed one upon the other, so that the point of the one angle coincides with the point of the other, and the lines that form the one angle coincide *in direction* with the lines that form the other.

Def. E. Of two angles one is larger than the other, when a line may be drawn between the lines forming the first angle, from the point where they meet, so as to form with either of them an angle equal to the second angle.

Def. X. When one straight line meets another, and makes the adjacent angles equal to each other, each of those angles is called a right angle, and the one line is said to be perpendicular to the other.*



Def. XI. An obtuse angle is an angle which is larger than a right angle.

Def. XII. An acute angle is an angle which is less than a right angle.

Def. XV. A circle is a plane bounded figure, contained by one line, which is called the circumference, and is such, that every point in it is at the same distance from a certain point within, which is called the centre.

Def. F. A straight line drawn from the centre to a point in the circumference, is called a radius of the circle.

It is another mode of stating the definition of a circle to say, that all the radii of the same circle are equal to one another.

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* Beginners are extremely apt to confound *perpendicular* with *straight;* and also to suppose that if one line be perpendicular to another, the latter must be *horizontal.* This is not at all necessary: the lines may be in any conceivable position, so long as their inclination to each other is such as is stated in the definition.

Def. XVII. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

It is obvious that the half of a diameter is a radius.

Def. XXI. A triangle is a plane bounded rectilineal figure, formed by three straight lines.

Def. XXII. A quadrilateral is a plane bounded rectilineal figure, formed by four straight lines.

Def. XXIII. A polygon is a plane bounded rectilineal figure, formed by more than four straight lines.

Def. XXIV. An equilateral triangle is a triangle which has its three sides equal to one another.

Def. XXV. An isosceles triangle is a triangle which has two sides equal. (An equilateral triangle may be regarded as an isosceles triangle if only two of the sides are considered.)

Def. XXVI. A scalene triangle is a triangle no two sides of which are equal.

Def. XXVII. A right-angled triangle is a triangle one of whose angles is a right angle. That side of a right-angled triangle which is opposite to the right angle, is called the hypotenuse.

Def. XXVIII. An obtuse-angled triangle is a triangle one of whose angles is obtuse.

Def. XXIX. An acute-angled triangle is a triangle which has all its angles acute.

Def. XXXV. Parallel right lines are lines drawn in the same plane, which, however far they may be produced both ways, do not meet.

Def. G. A parallelogram is a four-sided figure, the opposite sides of which are parallel.

Def. H. A rectangle is a parallelogram which has all its angles right angles.

Def. XXXI. A square is a rectangle which has all its sides equal. (Or, a square is a four-sided figure, with all its sides equal, and all its angles right angles.)

Def. XXXIII. A rhombus is a parallelogram which has all its sides equal, but its angles not right angles.

Def. XXXIV. A trapezium is a four-sided figure, which has two of its sides parallel.

POSTULATES.

The figures reasoned about in geometry are *mental* figures—figures which we imagine to ourselves. Diagrams drawn on paper are used only to assist the imagination and memory. With practice the mind may be trained to go through demonstrations of considerable length without any diagram before the eyes. There would therefore be nothing absurd or inconsistent in assuming ourselves at once to be able to construct any required figure, such as a parallelogram, or an equilateral triangle, or to make one line perpendicular to another, without the aid of any simpler processes; that is to say, we might at once *imagine* any figure to be before us, and then reason about its properties. But part of the science of geometry consists in showing how the simplest figures that we can conceive may be combined so as to form such as are more complicated. We must, however, assume, that we are able at once to make those elementary figures, by combinations of which all other figures may be produced. Accordingly, in geometry, the following postulates (or "things demanded") are laid down.

First postulate. Let it be granted, that a right line may be drawn from any given point to any other given point.

Second postulate. Let it be granted, that a given finite right line may be produced (or continued) both ways to any length that we please.

Third postulate. Let it be granted, that a circle may be described with any given point as centre, and at any distance from that centre.

It is obvious, that no combination of simpler figures is either requisite or possible, in order to produce the "construction"* specified in these postulates.

The most important aspect of the postulates is, however, that they are of a *restrictive* force. It is implied in them, that no construction is allowable in geometry which cannot be effected by combinations of these three primary constructions.

AXIOMS.

The reasonings of geometry are based upon certain primary truths with respect to magnitudes and lines, which are self-evident, and do not admit of being demonstrated by the application of truths of a simpler kind. These primary propositions are called axioms. They are the following:—

I. Magnitudes which are equal to the same, are equal to one another.

II. If equals be added to equals, or if equals be added to the same, the sums will be equal.

III. If equals be taken from equals, or if equals be taken from the same, the remainders will be equal.

IV. If equals be added to unequals, the sum of the one of the equals and the greater of the unequals will be greater than the sum of the other equal and the smaller of the two unequals.

V. If equals be taken from unequals, the remainder left, when one of the equals has been taken from the greater of the unequals, will be greater than the remainder left when the other equal has been taken from the smaller of the unequals.

A. If the larger of two unequal magnitudes be added to the larger of two other unequal magnitudes, and the less to the less, the sum of the two larger magnitudes will be larger than the sum of the two less.

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* The process by which any figure is produced, is called the construction of the figure.

VI. Things which are doubles of the same, or of equals, are equal to one another.*

VII. Things which are halves of the same, or of equals, are equal to one another.†

That "the whole is greater than its part," need not be set down as a separate proposition, since it is involved in the very import of the words *whole* and *part*.

That magnitudes which coincide with each other, that is, which fill exactly the same space, are equal (or of the same size), is a mere *verbal* truth; "to fill the same space," is "to be of the same size." "To be of the same size," however, does not always involve "to fill the *same* space." Areas may be equal, and yet of different shapes.

X. Two straight lines cannot enclose a space.

B. Two straight lines may be so placed one on the other as to coincide with each other in direction, and cannot diverge from one another if they coincide for any portion of their length.‡

XII. If two straight lines meet a third, and make the two interior angles on the same side of it together less than two right angles, those lines will meet if produced.

C. If part of one bounded figure (and part only) be inside another bounded figure, the peripheries of those two figures must intersect one another in two points at least.

---

* This is properly only a particular case of the second axiom.

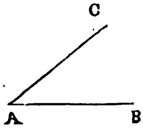
† The first part of this axiom is only the definition of *halves* in another form. The second part is properly an inference from Axiom 6. For if the halves of two magnitudes were unequal, the doubles of those halves, that is, the magnitudes themselves, would be unequal.

‡ The latter part of this axiom is commonly set down as a corollary to the thirteenth proposition; but it is really made use of much earlier,—in the fourth proposition, in the proof of which we have to make use of the truth that two equal straight lines will coincide throughout their entire length.

That all right angles are equal, is a proposition that admits of being proved.

THE NAMING OF ANGLES.

An angle is named either by a letter placed at the point of intersection of the lines which form it, or by three letters, one at the point of intersection of the lines forming it, the others at some point of these two lines respectively.* In reading these three letters, that letter is always placed between the other two which is at the point of the angle. Thus the annexed angle would be called either B A C or C A B, but not A B C or A C B.



SUPERPOSITION OF FIGURES.

A device of which frequent use is made in proving propositions in geometry, is to compare figures, by supposing one of them removed from its separate position and applied to the other, so as to coincide with it, wholly or partially, as the case may be. No difficulty will be found in this, if it be remembered that the figures with which we are really concerned, exist only in the mind, and have no definite relation to any particular locality, so that they may just as well be conceived of in one position as in another, with all their parts and properties exactly the same as before. When,. therefore, we talk of applying one figure to another, we mean that we are to conceive the figures in our minds as being so situated, with respect to each other, that certain parts, or all the parts of each may coincide

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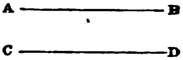
* A triangle is named by placing a letter at each of the three angles. Thus it comes to pass that the same three letters by which each of the angles of the triangle is named, are also used to name the *whole triangle itself.* One unfortunate consequence of this is that beginners often get into confusion, and when they name an *angle* of a triangle (that is, the opening formed between *two* of the sides of the triangle), fancy that they are talking about the *whole of the three-sided figure.* Many teachers would be surprised to find how often this blunder is made without being detected. It needs to be guarded against with the most constant vigilance. In order that the eye may help the understanding, the mark ∠ will be used in this book for the word *angle*, and the mark △ for the word *triangle.* Of course ∠s will mean *angles*, and △s will mean *triangles.*

In doing this there are some mistakes against which beginners must be particularly cautioned. They may seem trifling, but they must not be disregarded. Not the smallest point must be taken for granted, if it is not one of the facts which we are supposed to be acquainted with, before we begin our reasoning in any case, or some truth which has been rigidly demonstrated. It is this *exactness* which is demanded, which gives to the study of geometry its great value. The reasoning applied is in no respect different in kind from what is applied to any other subject: but the subject-matter of the science is perfectly simple; there are no collateral matters to be *taken into account;* and every one can form a perfect conception of the things reasoned about. Respecting the meaning of equal and unequal, straight line, circle, &c., there can be no mistake. But this, at the same time, occasions a danger to which beginners are liable. Many of the truths demonstrated are so very evident, that the mind gets impatient at going through the steps of the proof. It must be borne in mind, that the object of the propositions is not *to convince us* of the truth of certain facts, or of the existence of certain properties in figures, but *to show in what way these facts or properties are to be deduced from the simple elements to which we are limited at starting.* There must not, therefore, be a single break or flaw in the chain of demonstration which connects the first beginning with the final conclusion. The most minute exactness must be observed at every step.



Suppose that we have two right lines, A B and C D, of equal length, and that it is requisite for any purpose that the one should be supposed to lie upon the other. We may say, "Place the point A upon the point C, and let the line A B lie along the line C D; then, because the lines are equal, the point B will coincide with the point D." But it would be wrong to say, "Let the point A be placed upon the point C, and the point B upon the point D," or anything equivalent to this, because that would imply, that, after we have fixed the position of the point A, and the direction of the line A B, we can then *make the point B fall where we like.* When we have fixed the position of one end of a given line, and the direction in which it is to lie, we have nothing more under our control. *The length of the line itself determines where the other extremity will fall.* We may be able to say where that other end *will* fall, but we must not use language implying that we can make it fall where we choose.

Suppose that there are two ∠s of the same size,[*]

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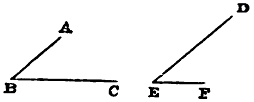
[*] Let it not be forgotten that the size of the ∠s in no way depends upon the length of the lines which form them. The ∠ A B C would be of the same size, whether contained by lines a mile long, or by lines of the length of a hair's-breadth.

A B C and D E F, and that the one ∠ has to be placed upon the other. We may say, "Place the point B upon the point E, and let the line B C lie along the line E F (whether the point C will fall on the point F depends upon whether the line B C is equal to the line E F or not). But having done that, we must not say, let (or make) the line B A lie along the line E D; because that would imply that when we have fixed the position of one of the lines containing a given ∠, we can also make the other line lie in any position that we choose. But this we cannot do. When one side of a given ∠ is fixed, it is *the size of the angle* which determines in what direction the other side will lie. It is obvious, that if the ∠s A B C and D E F are equal, and the sides B C and E F coincide in direction, the sides B A and E D will also coincide in direction (if the ∠s be made to lie on the same side of the coinciding lines). But that is not because we make them coincide, but because the equal size of the ∠s makes them do so.*



PROBLEMS AND THEOREMS.

The propositions in the First Book of Euclid are divided into two classes, problems and theorems. In a problem, it is required to show how a figure, having certain specified properties, may be constructed by a combination of those elementary constructions to which we are limited by the postulates. In a theorem it is required to demonstrate by the direct or indirect application of those elementary truths which are laid down in the axioms that figures having certain specified properties have also certain other properties. (Some remarks, in further explanation, will be most conveniently appended to the second and fifth propositions.)

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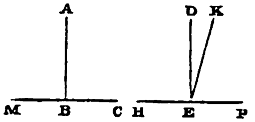
* It may seem almost trifling to introduce such simple considerations as these so elaborately; but I have known persons who, after studying Euclid for years, have been unable to demonstrate properly the fourth proposition in the first book, from not having attended to the caution above laid down.

Let it be observed, that the object of the problem is not to furnish us with practical rules for drawing figures on paper, &c., having the required properties (with such an approach to accuracy as the nature of the means employed would admit of), for, by the aid of compasses, and other mathematical instruments, this may be effected in a more convenient manner. Nor are they designed to help us in *conceiving* of the figures which it is their object to produce. For, if we wish to have a square, or an equilateral △, to reason about, there is nothing to hinder us from *conceiving* the figure at once. The object of the problems is purely that described in the preceding paragraph. Again, it is not the object of the theorems to *convince* us of the truth of what is proved in them. In a large number of cases, most people would never imagine it possible to doubt the truth of these propositions. The object is *to show by what intermediate steps the truth to be proved must be connected with the primary axioms and definitions with which we start.* Most beginners are surprised and tired by what appears to them the unnecessary tediousness of some of the demonstrations. This is only because they do not understand the real object of those demonstrations.

All right angles are equal.

This may be proved as follows:

Let A B C and D E F be any two right ∠s. Produce the line C B to any point M, and the line F E to any point H.



Suppose the figure M A B C to be applied to the figure H D E F, so that the point B may coincide with the point E, and the line M C lie along the line H F.

Let the figures lie on the same side of the common line:

Then the line B A must lie along the line E D; for if it do not, but lie in some other direction, such as E K, inasmuch as H E K and K E F would be the same as the ∠s M B A and A B C, we should have the ∠s H E K and K E F equal to one another (see the definition of right ∠s); and also, H E D and D E F equal to one another.

But as the ∠ H E K is larger than the ∠ H E D, it must be larger than the ∠ D E F, which is equal to H E D.

But D E F is larger than K E F; much more, therefore, must H E K be larger than K E F, or H E K and K E F are unequal to each other. But this contradicts what was before known, that H E K and K E F, or (what are the same ∠s,) M B A and A B C are equal to each other.

Now a supposition, which of necessity leads to a contradiction of what is known to be true, must be a false supposition.

The supposition which led us to the above contradiction was, that the line B A did not coincide with the line E D when the figures were placed upon one another as described.

This supposition, then, is a false one. But, to show that it is false to suppose that B A and E D do not coincide, is equivalent to proving that it it true that they do coincide, or, in other words, that the ∠ A B C is equal to the ∠ D E F.

PROPOSITION I.

Problem.

On a given finite right line to construct an equilateral triangle.*

The following postulates and axioms are made use of in the construction and demonstration in this proposition :—

1. *Postulate I.* A straight line may be drawn from a given point to any other given point.
2. *Postulate III.* A circle may be described with a given centre, and at any distance from that centre.
3. *Axiom I.* Magnitudes which are equal to the same, are equal to one another.
4. *Axiom C.* If part of one bounded figure (and part only) be inside another bounded figure, the boundaries of those two figures must intersect one another in two points at least.
5. *Definition F.* All radii of the same circle are equal to one another.

Let A B be the given finite right line.

With the centre A, and at the distance A B, describe the circle C B E. (Post. III.)

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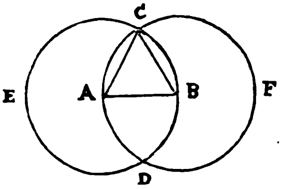
* By speaking of the line as "given," it is meant that its position is given. By speaking of it as "finite," it is meant that its length is given. A △ is said to be constructed on a given finite right line when that line forms one of the sides of the △. Respecting the use of the marks △ and ∠, see the note on p. 9.

Caution.—Whenever any *construction* is required in the course of a proposition, the learner should never be allowed to draw the whole figure to begin with, but should be made to draw it *step by step*, as directed in the text. This is of vital importance.

With the centre B, and at the distance B A, describe the circle A C F. (Post. III.)



Part of the circle E C B is inside the circle A C F, and part outside, therefore these circles must intersect in two points. Call these points C and D.

Take either of these points (say C), and join A and C by the right line A C,* and C and B by the right line C B. (Post. I.)

We thus get a △ A C B, and that this △ is a △ such as was required, may be proved as follows:—

The lines A B and A C are radii of the same circle E C B, and therefore are equal to one another (Definition F); and the lines B C and B A are radii of the same circle A C F, and therefore are equal to one another.

But since the lines A C and B C are both equal to the line A B, they are equal to one another; or, in other words, A C B is an equilateral △, and is constructed on the line A B.

It would have done equally well if we had taken the point D, and joined D and A, and D and B. The demonstration would have been word for word the same as above, only substituting D for C.

PROPOSITION II.

From a given point to draw a straight line equal to a given finite straight line.

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* Two points are said to be *joined* by a right line when a right line is drawn from one to the other.

For the construction in this proposition we must be able—

1. To join two given points by a right line. (Post. I.)
2. To produce a given right line to any length. (Post. II.)
3. To describe a circle with a given centre and at a given distance from that centre. (Post. III.)
4. On a given finite right line to construct an equilateral △.* (Prop. I.)

In proving that the construction furnishes us with a line drawn as required, we must know—

1. That radii of the same circle are equal to one another. (Def. XIV.)
2. That magnitudes which are equal to the same, are equal to each other. (Axiom I.)
3. That if equals be taken from equals, the remainders are equal. (Axiom III.)

Let A be the given point, and B C the given right line.

Join A and one end (say C) of the line B C, by the line C A. (Post. I.)

On C A construct an equilateral △ A D C. (Prop. I.)

With C as centre, and at the distance C B, describe a circle. (Post. III.)

Produce the line D C till it cuts the circumference of this circle in the point E. (Post. II.)

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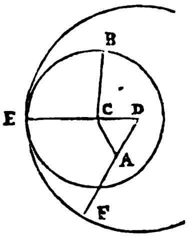
* In the construction, in propositions, use is frequently made of complex constructions, the mode of performing which has been shown in some previous proposition. Strictly speaking, all the details of these constructions ought to be gone through for the case in hand; but, to save time and avoid confusion, it is usual simply to *suppose* that these details have been gone through, and that we have got the required result. In any given case, it would be possible to analyse the construction, however complex, into its ultimate elements, namely the postulates.

With D as centre, and at the distance D E, describe a circle. (Post. III.)



Produce the line D A to meet the circumference of this circle in the point F. (Post III.) Then A F will be a line drawn as required. For—

The lines D E and D F are equal, being radii of the same circle. (Def. XIV.)

The lines D C and D A are also equal, being sides of an equilateral △.

If the equals, D C and D A, be taken from the equals, D E and D F, the remainders, C E and A F, will be equal. (Axiom III.)

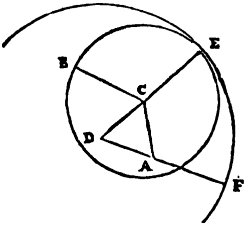
The lines, C B and C E, are equal, being radii of the same circle. So that the lines C B and A F are both equal to the line C E.

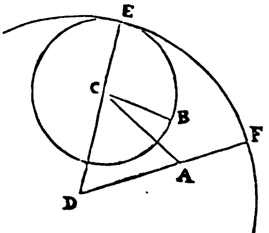
Therefore C B and A F are equal to each other. (Ax. I.) And as the line A F is drawn from the point A, and is equal to B C, it is drawn as was required.

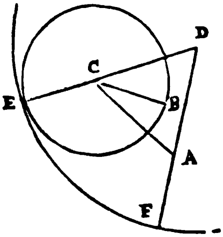
Having got one line drawn from the given point, and of the given length, we can draw a line of the same length from the same point *in any other direction.* Any line drawn from the point A to a point on the circumference of a circle drawn with A as a centre, and at the distance A F, will be of the same length as A F.

Beginners always find this proposition very troublesome to remember. The principle upon which the construction is made must be carefully attended to. The given point may be joined with either extremity of the given line. The equilateral △ constructed on this joining line, may be constructed on either side of it. The circle described with the given line as a radius, must have that end of the given line as a centre, which was joined by a right line to the given point. The centre with which the second circle is described must be that vertex of the equilateral △ which is opposite to the line on which it was described. The radius with which the second circle is to be described is to be obtained by producing that side of the equilateral △ which lies between this vertex and the end of the given line which was joined with the given point, until it cuts the circumference of the first circle. Then that side of the equilateral △ which lies between the centre of the second circle and the given point, must be produced through the given point (not backwards through the centre of the second circle), till it meets the circumference of the second circle.

As there are two ends to the given finite right line, with either of which the given point may be joined, and as the equilateral △ may be constructed on either side of the joining line, there are four distinct constructions by which the problem may be solved in any given case. Thus, for the relative positions of the line and point given above, besides the construction depicted before, there are the three following:—







The letter B has been put where the letter C was, and C where B was, in the third and fourth cases, which allows the description of the construction, and the proof of its correctness, to remain word for word the same in all four cases.

When the proposition has been mastered with the first construction, it should be carefully gone through with each of the other three; and then all four methods should be applied to several cases where the given line and the given point are in different relative positions. By attending to the *general* directions, all possibility of mistake will be avoided.

In repeating the proposition, let care be taken not to use the same letters to mark the different points of the figure.

PROPOSITION III.

From the greater of two given straight lines, to cut off a part equal to the less.

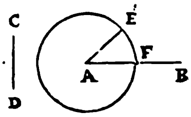
To do this we must be able,—

1. From a given point to draw a straight line equal to a given straight line. (Prop. II.)
2. With a given centre, and at a given distance from that centre, to describe a circle. (Post. III.)

To prove that the construction is correct, we must know,—

1. That radii of the same circle are equal. (Def. F.)
2. That magnitudes which are equal to the same are equal to each other. (Ax. I.)

Let A B and C D be the two given straight lines, of which A B is the greater.



From the point A draw the straight line A E equal to C D.* (Prop. II.)

With A as centre, and at the distance A E, describe a circle, cutting the line A B in the point F. (Post. III.) A F is the part of A B which was required.

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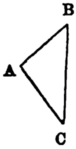
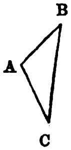
* At this step the whole of the construction given in

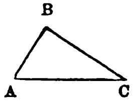
For, A E and A F are equal, being radii of the same circle (Def. XIV.); and as A E and C D are equal, A F and C D are both equal to A E. (Ax. I.)

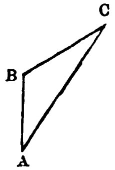
Therefore A F and C D are equal to one another; or, in other words, A F has been cut off from A B of the same length as C D.

The shape of a △ is not determined if we know only the lengths of two of its sides. For an infinite number of △s may be constructed, having two sides of the same length respectively. Thus, in the following △s, the sides A B and B C are of the same length respectively in each :—









It is obvious, therefore, that we cannot pronounce two △s to be of the same shape *merely* because two sides of the one are equal to two sides of the other, each to each; for if we could, we should be able to say that all the above △s were of the same shape. But if we also know the size of the ∠ between A B and B C, then the shape of the △ is determined. Let the above caution be carefully attended to. The neglect of it is a very common error with beginners.

Prop. II. is *supposed* to be gone through, and the line A E to be a line obtained as the *result* of all the operations there indicated. Without going through the construction we cannot tell whether the line required will fall exactly in the position A E; but as the *direction* of the line A E is not a point of the least consequence for the rest of the construction, we need not trouble ourselves about it.

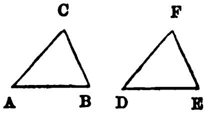
PROPOSITION IV.

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angle included between those two sides in the one equal to the angle included between the corresponding sides of the other, then the bases, or third sides, will also be equal, and the remaining angles will be equal, each to each, those being equal to one another to which equal sides are opposite.

In proving this proposition we are supposed to be able to place a given figure in any other position that we choose (*see* p. 9); and we are supposed to know—

1. That two straight lines may be so placed one upon the other as to coincide with each other in direction, and cannot diverge from one another if they coincide for any portion of their length. (Ax. X.)
2. That two straight lines cannot enclose a space. (Ax. XI.)

Suppose that A B C and D E F are two △s, in which the sides A C and C B of the one are equal to the sides D F and F E of the other, each to each (*i.e.*, A C to D F, and C B to F E), and the ∠ A C B (included between the sides A C and C B) equal to the ∠ D F E (included between the sides D F and F E).*



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* Let it be observed, that this is all that we are able to assert about the △s, *to start with.* Although it is true that all the other parts are also equal to each other, yet this has still to be proved, and before it has been proved we must on no account take it for granted, or speak of the equality of the other parts as a truth already known.

It has now to be proved that the side A B is equal to the side D E; that the ∠ C A B is equal to the ∠ F D E; that the ∠ C B A is equal to the ∠ F E D; and that the △s are equal in area.

The mode of proving this is to show that the one △ may be so placed upon the other as to coincide with it in every part.

Apply the △ A C B to the △ D F E, so that the point A may coincide with the point D, and the straight line A C lie along the straight line D F. (Ax. X.) Then, because the lines A C and D F are equal to one another, the point C will coincide with the point F.*

Next, let the △s lie on the same side of the line D F. Then, because the ∠ A C B is equal to the ∠ D F E, the line C B will lie along the line F E; † and because the point C coincides with the point F, and the line C B lies along F E, and is equal to it, the point B will coincide with the point E. ‡

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* You must not say, "*Let* the point C coincide with the point F;" because that would imply that you could let it coincide with it, or not, as you pleased. When one end, A, has been fixed, and the position of the line A C determined, the length of the line determines where the other end will fall. See p. 11. It will do just as well if you place the point C upon the point F, and let the line C A lie along the line F D. Then because the lines C A and F D are equal, the point A will coincide with the point D. Either way leads us to the result that the line A C will coincide exactly with the line D F.

† You must not say, "*Let* the line C B lie along the line F E;" for that would imply that you could *let* or *make* it fall in some other direction if you chose. When the position of one of the lines that contain an ∠ is fixed, it is *the size of the ∠ that determines in what direction the other side will fall.* See p. 12.

‡ Beginners often make the mistake of saying that the

It was shown before that the point A coincides with the point D. As the point B also coincides with the point E, it follows that the straight line A B lies along the straight line DE; otherwise these two straight lines would enclose a space, which is impossible. (Ax. XI.)

Thus it has been shown that the △ A B C may be placed upon the △ D E F in such a manner that it will coincide with it in every part. In other words, it has been shown that the side A B is equal to the side D E; that the ∠ C A B is equal to the ∠ F D E; that the ∠ C B A is equal to the ∠ F E D; and that the △s are equal in area.

Instead of first placing the line A C on the line D F, and then showing that because the ∠s A C B and D F E are equal, the line C B will lie along the line F E, it would have done just as well if we had placed the line C B on the line F E, and then shown that the line C A will lie along the line F D. But it is necessary to begin by making two of those sides coincide, which we know at starting to be equal. It would be of no use to begin by placing the point A upon the point D, and letting the line A B lie along the line D E; because, if we did so, we should be unable to show that the line A C would lie along the line D F; for, we have no right to assert that the ∠ B A C is equal to the ∠ E D F, until we have proved that it is so.

When the proposition has been mastered with the figure given above, let it be supposed that, instead of knowing, to start with, that the sides A C and C B are equal respectively to D F and F E, and the ∠ A C B to the ∠ D F E, we know

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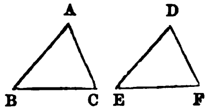
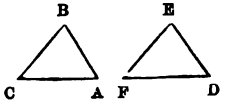
point B will coincide with the point E, because the ∠ A C B is equal to the ∠ D F E. The size of the ∠ which one line makes with another can only determine the *direction* in which the lines lie with respect to each other. We must know the length of the lines as well, in order to be able to say where the *ends* of them will fall.

that the sides C B and B A are equal respectively to the sides F E and E D, and the ∠ C B A to the ∠ F E D. By changing the position of the letters to the following,



the proof may be gone through word for word as before.

Then (in the original figure) let it be supposed that we know that the sides C A and A B are equal respectively to F D and D E, and the ∠ C A B to the ∠ F D E. By altering the position of the letters to the following, the proof remains word for word the same:—



Then let △s of a different shape be employed, and different letters placed at the corners.

PROPOSITION V.

The angles at the base of an isosceles triangle are equal; and if the equal sides be produced, the angles at the other side of the base are likewise equal.

An isosceles △ is a △ which has two sides equal. In this proposition, therefore, we are supposed to know, to start with, that two sides are equal, and this is all that we know about the △ to begin with. What we have to prove is, that, if that be the case, then the ∠s opposite the equal sides are equal to one another, and that if the equal sides be produced, the ∠s at the other side of the base are equal to each other.

For the construction used in proving this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)

2. To cut off from the longer of two given straight lines a piece equal to the shorter. (Prop. III.)

For the proof of the proposition we must know,—

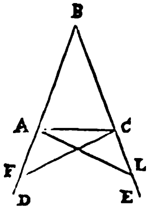
1. That if equals be taken from equals, the remainders are equal. (Ax. III.)

2. That if two △s have two sides of the one equal to two sides of the other, each to each, and have likewise the ∠s contained by these two pairs of sides equal, then the △s will also be equal in every other respect. (Prop. IV.)

Suppose A B C to be a △, having the sides A B and B C equal to each other. We have to prove that the ∠s B A C and B C A are equal to each other, and likewise that the ∠s F A C and L C A are equal to each other.

To prove this proposition, the following construction is made:—

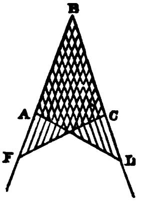
In B A produced take any point, (as F), and from B E cut off a part, B L, equal to B F (Prop. III.) Join F and C by the right line F C, and A and L by the right line A L. (Post. I.)



The proof of the first part of the proposition (namely that the ∠ B A C is equal to the ∠ B C A) consists of three main steps. First, it is shown that the ∠ B A L is equal to the angle B C F. Secondly, it is shown that the ∠ C A L is equal to the ∠ A C F. Thirdly, it is inferred that if the equal ∠s C A L and A C F be taken from the equal ∠s B A L and B C F, the remainders, namely the ∠s B A C and B C A, will be equal.

1. To prove that the ∠ B A L is equal to the ∠ B C F.

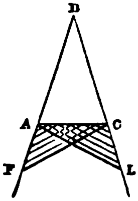
There are two △s, B A L and B C F (shaded respectively by parallel lines), which overlap each other (where the lines of shading cross).



To prove that the *angles* B A L and B C F are equal to each other, it is necessary to show that the *triangles* B A L and B C F are equal to each other in every respect.

Now we know that the side A B of the first is equal to the side B C of the second; and the side B L of the first was made equal to the side B F of the second. Moreover, the ∠ included between the sides A B and B L of the first, is *the same* as the ∠ included between the sides C B and B F of the second. So that, in these two △s, two sides and the ∠ between them in the one, are equal respectively to two sides and the ∠ between them in the other. Therefore (according to the fourth proposition) the two △s are also equal in all other respects. Therefore, the ∠ B A L is equal to the ∠ B C F;[*] the ∠ B L A is is equal to the ∠ B F C; and the side A L is equal to the side F C.

2. To prove that the ∠ C A L is equal to the ∠ A C F :—



These ∠s are parts of the *triangles* C A L and A C F, which overlap one another. To prove that the *angles* C A L and A C F are equal, it is necessary to prove that the *triangles* C A L and A C F are equal in every respect.

Because B F is equal to B L, and B A is equal to

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[*] We have thus reached the first main step of the proof.

B C, the remainders A F and C L are equal (Ax. 3). Moreover, the line A L was proved equal to the line C F, and the ∠ C L A (which is the same as B L A) to the ∠ A F C (which is the same as B F C), when the △s B A L and B C F were shown to be equal in every respect. So that in the two △s C L A and A F C, the two sides C L and L A of the one are equal respectively to the two sides A F and F C of the other, and the included ∠ C L A of the one, is equal to the included ∠ A F C of the other. Therefore (according to the fourth proposition) these two △s are also equal in all other respects. Among these is that the ∠ C A L is equal to the ∠ A C F.

3. If from the equal ∠s B A L and B C F, the equal ∠s C A L and A C F be taken away, the remainders, namely the ∠s B A C and B C A, will be equal to each other (Ax. 3). But these are the ∠s opposite the equal sides of the isosceles △ A B C. Therefore the first part of the proposition has been proved.

The second part has also been proved. For the ∠s at the other side of the base, namely F A C and L C A, are parts of the △s A F C and C L A, which have been proved to be equal in every respect.*

Cor. All the ∠s of an equilateral △ are equal.

The proof of the next proposition is of the kind called *indirect.* An indirect proof is of the following nature:— If we know that out of any number of assertions, some one must be true; and if we know, or can prove, that all but one are false, we are then able to affirm that the remaining one must be the true one. If, for example, it could be demonstrated that one or other of three men—A, B, and C—committed a certain crime, and we knew or could prove, that neither A nor B committed it, it would follow that C must have committed it. Of the propositions, A is equal to B, A is less than B, A is greater than B (A and B being two magnitudes that admit of comparison); it is clear that one or other must be true. If we can show that A is not equal to B, and also that A is not less than B, it follows that A is greater than B.

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* When this proposition has been mastered as given above, it should next be proved without the aid of the accessory figures; then with different letters in the diagram; and then with figures of different shape, and drawn in different positions.

In order to prove that a proposition or assertion is false, we may either prove that something which contradicts it is true, or we may show that some falsity or absurdity would be a necessary consequence of it. Now it is assumed as an axiom, that "*nothing which is absurd or false can be a necessary consequence of anything that is true.*" If a false assertion be a necessary consequence of some other assertion, that other assertion must itself be false. When an assertion or proposition is demonstrated to be false, by showing that if we admitted it we should be compelled also to admit something false or absurd, the proof is called *reductio ad absurdum* (Reduction to an absurdity). Proofs of this kind are very common in mathematics. Some people have a fancy that they are not so cogent or conclusive as direct proofs; but this is altogether a mistake.

PROPOSITION VI.

If two angles of a triangle are equal, the sides which are opposite the equal angles are also equal.

This proposition is the converse of the fifth. In the fifth we started by supposing that we knew two sides of a △ to be equal, and proceeded to prove that the ∠s opposite those equal sides are also equal. In the sixth we are not supposed to know anything about the sides of the △, to start with, but to know something about two *angles* of the △—namely, their equality. The equality of the sides has to be demonstrated.

The proof of this proposition is of the indirect kind. Instead of its being shown at once that the two sides are equal, it is shown that they cannot be unequal.

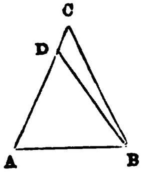
To prove this proposition we require to be able,–

1. To draw a straight line from one given point to another. (Post. I.)
2. To cut off from the greater of two given lines a part equal to the less. (Prop. III.)

And we require to know,—

1. That a part of any whole is less than that whole.
2. That if, in two △s, two sides of the one are equal to two sides of the other, each to each, and the included ∠ of the first △ also equal to the included ∠ of the second △, then those two △s are equal in every respect.

Let A B C be a △, with respect to which it is known that the ∠ C A B is equal to the ∠ C B A. We have to prove that the side A C is equal to the side C B.



This is proved by showing that A C and C B are not *un*equal.

That A C and C B *are not* unequal is proved by showing that it leads to an absurdity to suppose that they *are* unequal.

Suppose the side A C were not equal to the side C B; and suppose, first, that A C were greater than C B,

We should then be able to cut off from A C a part, A D, of the same length as C B.

Suppose that this were done, and that the points D and B were joined by the straight line D B, We should then have two △s, D A B and C B A, in which the side D A would be equal to the side C B, and the side A B common to the two △s, and the ∠ D A B equal to the ∠ C B A (we were supposed to know this last point to begin with), so that the two △s, D A B and C B A, would have two sides of the one (namely, D A and A B), equal to two sides of the other (namely, C B and B A), each to each, and the included ∠ (D A B) of the first △ equal to the included ∠ (C B A) of the second. Therefore, according to the fourth proposition, these two △s would be equal to each other in every respect, and among others, in *area.*

But the △ D A B is only a part of the △ C B A.

Therefore, if we were to suppose that, while the ∠ C A B is equal to the ∠ C B A, the side A C could be longer than C B, we should be compelled to admit that a part could be equal to the whole of which it is a part.

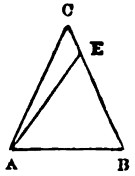
Now, a supposition which leads to an absurdity must be itself absurd.

Therefore it is absurd to suppose that A C could be longer than C B, while the ∠s C A B and C B A are equal.

Next, suppose that the side B C were longer than A C,

We should then be able to cut off from B C a part, B E, equal to A C.

Suppose that this were done, and that the points A and E were joined by the right line A E,



We should then have two △s C A B and E B A, in which the sides C A and A B of the one would be equal to the sides E B and B A of the other, each to each, and the included ∠ C A B of the first, equal to the included ∠ E B A of the other.

Therefore, according to the fourth proposition, these △s would be equal in every respect, and, among others, in area.

But the △ C A B is only a part of the △ E B A.

Thus it appears that if we were to suppose that the side B C is longer than the side A C, while the ∠ C A B is equal to the ∠ C B A, we should be compelled to admit that a part is equal to the whole of which it is a part.

But a supposition which leads to an absurdity must be itself absurd.

Hence it is absurd to suppose that B C is longer than A C, while the ∠ C B A is equal to the ∠ C A B.

It has thus been proved that A C cannot be greater than B C, and also that B C cannot be greater than A C—*i. e.*, that A C and B C are not unequal.

But A C and B C must be either equal or unequal; therefore, as they are not unequal, they must be equal.

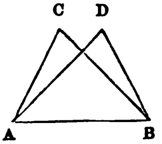
(Let this proposition be gone through next with the figure in a different position, and then with the aid of different letters to mark the sides, &c.)

Cor. It follows from this that all the sides of an equi-angular △ are equal.

PROPOSITION VII.

Upon the same base, and on the same side of it, there cannot be two triangles having their sides which are terminated in the one extremity of the base equal to one another, and likewise those that are terminated in the other extremity of the base equal to one another.

That is, supposing that there were two △s such as A C B and A D B, on the same base A B, and on the same side of it, then if the two sides A C and A D, terminating at A, were equal to each other, the sides B C and B D, terminating at the other extremity of the base, could not also be equal to each other; or if B C and B D were equal, then A C and A D could not also be equal to each other. It is not meant that *neither* pair of conterminous sides can be equal, but that *both* pairs cannot be equal at the same time.*



This proposition is proved by showing that if we were to suppose it possible that there could be two such △s as are described in the enunciation, we should be compelled to admit something to be true which is obviously impossible.

For the proof of the proposition we require,—

1. To be able to draw a straight line from one given point to another. (Post. I.)
2. To produce a given finite right line to any length. (Post. II.)

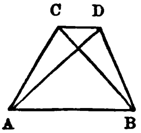
We must also know,—

1. That if a magnitude be less than one of two equal magnitudes, it is also less than the other.
2. That if a magnitude be less than the smaller of two unequal magnitudes, it is also less than the larger.
3. That if two sides of a △ are equal, the ∠s

---

* The beginner must not suppose that an elaborate proof of this proposition is unnecessary, because the inequality of one or other pair of sides is obvious to the eye. Geometrical truths are never to be settled by this test. And, in fact, if a scale sufficiently large were adopted, it would be possible to draw two △s placed like those in the diagram, such that the inequality of the sides could scarcely be detected by the most accurate measurement.

opposite to them are also equal, and if the equal sides be produced, the ∠s on the other side of the base are likewise equal (Prop. V.).



Let us suppose that there could be two △s A C B and A D B, on the same base A B, and on the same side of it, and such that A C and A D are equal to one another, and also B C and B D equal to one another.[*] First let the △s be such that the vertex of each falls outside the other.

Join the points C and D by the right line C D.

In the △ A C D, if A C and A D be equal to one another, the ∠ A C D is equal to the ∠ A D C (Prop. V.)

If the sides B C and B D could also be equal to one another, then also in the △ B C D the ∠ B C D would be equal to the ∠ B D C.

But the ∠ B C D is less than the ∠ A C D (being a part of it).

Therefore the ∠ B C D must also be less than the ∠ A D C, which is equal to the ∠ A C D.

But the ∠ A D C is less than the ∠ B D C (being a part of it).

Therefore the ∠ B C D, which is less than the ∠ A D C, must also be less than the ∠ B D C.

But it was shown before, that if the sides B C and B D be equal, the ∠s B C D and B D C are equal to each other.

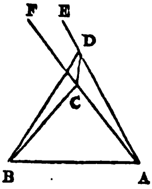
Thus the supposition that, in the △s A B C and A B D, the sides, A C and A D, could be equal to each other, and that the two sides B C and B D could also be equal to each other, renders it necessary that we should admit that two ∠s (namely B C D and B D C) are both equal and unequal at the same time, —which is an impossibility.

---

[*] The beginner must beware of supposing that the object of the proposition is to prove *that the sides* A C *and* A D *are not equal to the sides* B C *and* B D; a mistake which the writer has found made more frequently than might be supposed.

Now the supposition which leads us of necessity to this absurdity, must itself be absurd. That is, it is absurd to suppose that the two pairs of sides, A C and A D, and B C and B D, can be equal at the same time. (If one pair be equal, the other pair must be unequal.)

Next let us suppose it possible that there should be two △s on the same base and on the same side of it, having the sides that terminate at one extremity of the base equal, and likewise those that terminate at the other extremity of the base equal, so placed that the vertex of one of the △s (A B C) falls within the other △ (A B D).



Join the points D and C by the right line D C.

Produce the line A D to any point E, and the line A C to any point F.

If it were possible that the sides A D and A C should be equal to one another, and also the sides B C and B D equal to one another, then the △ D A C would be an isosceles △, and therefore the ∠s E D C and F C D, formed by producing the equal sides A D and A C, would be equal to one another; and, in the △ C B D, the ∠s B C D and B D C would be equal to one another.

But the ∠ B D C is less than the ∠ E D C (being a part of it): therefore the ∠ B D C is also less than the ∠ F C D, which is equal to E D C. Much more then is the ∠ B D C less than the ∠ B C D, which is greater than the ∠ F C D.

But it was before shown that, if we suppose the sides B C and B D to be equal to each other, as well as A D and A C, then the ∠ B D C must be equal to the ∠ B C D.

But it is impossible that the same ∠s (B D C and B C D) should be both equal and unequal.

Consequently, the supposition which led to this impossibility must be absurd. That is, it is absurd to suppose that, on the same base, and on the same side of it, there can be two △s, having the sides terminating in one extremity of the base equal to one another, and likewise those terminating in the other extremity of the base equal to one another.

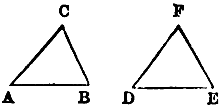
PROPOSITION VIII.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles will also be equal to each other in every other respect, that is, the angles of the triangles will be equal, each to each, and the triangles will be equal to each other in area.

For proving this proposition we must know,—

1. That two straight lines may be so placed one upon the other as to coincide in direction, and cannot diverge from one another if they coincide for any portion of their length. (Ax. B.)
2. That upon the same base, and upon the same side of it, there cannot be two △s having both pairs of conterminous sides equal. (Prop. VII.)

Let A B C and D E F be two △s, having the side A B equal to the side D E, the side A C to the side D F, and the side C B to the side F E.



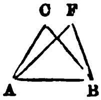
The proof of the proposition consists in showing that the two △s may be placed one on the other, so as to coincide in every part.

Place the point D on the point B, and let the line D E lie along the line A B.

Then, because the line D E is equal to the line A B, the point E will fall upon the point B.*

Let the △s lie on the same side of the base A B.

Then the line D F must lie along the line A C, and the line E F along B C, the point F coinciding with the point C. For if they did not, but lay in any other position, as, for example, in the annexed figure, we should have two △s, A C B and A F B, standing on the same base, and on the same side of it, and having the conterminous sides A C and A F equal, and likewise the conterminous sides B C and B F equal.



This was proved in the last proposition to be impossible. Consequently, the supposition which leads to this impossibility must itself be impossible. That is, it is impossible, when the side D E has been placed so as to coincide with the side A B, that the side D F should lie in any other position than along A C, and the side E F in any other position than along B C.

Therefore, as they must lie in some position or other, DF must lie along A C, and E F along B C.

Consequently, as the sides F D and D E coincide in direction with C A and A B, the ∠ F D E is equal to the ∠ C A B. In like manner the ∠ F E D is equal to the ∠ C B A, and the ∠ D F E to the ∠ A C B, and the area of the one △ is equal to the area of the other.

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* Beware of saying, "Place the point D upon the point A, and let the line D E lie along the line A B and *let the point* E *fall on the point* B." (See p. 9.)

It is indifferent which pair of sides be made to coincide to begin with.

The eighth proposition may be proved without having recourse to the seventh, as follows:—

We must know,—

1. That two straight lines may be so placed one on the other as to coincide in direction, and cannot diverge from one another if they coincide for any portion of their length. (Ax. B.)
2. That the sums or differences of two pairs of equal magnitudes are equal. (Ax. II. and III.)
3. That if two △s have two sides of the one equal to two sides of the other, each to each, and have likewise the ∠s included between those sides equal, the △s are also equal in all other respects. (Prop. IV.)
4. If two sides of a △ are equal to each other, the ∠s opposite to them are also equal. (Prop. V.)

FIG. 1.

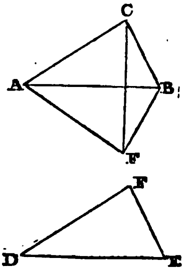
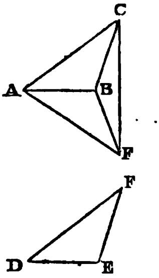


FIG. 2.



Let A B C and D E F be the two △s having the three sides of the one equal to the three sides of the other, each to each.

Let the △ D E F be applied to the △ A B C so that the point D may coincide with the point A, and the line D E lie along the line A B. Then because A B is equal to D E, the point E will coincide with the point B.

Let the △s lie on opposite sides of the common line A B.

Join the points C and F by the line C F. In the △ C A F, the side A C is equal to the side A F; therefore the ∠ A C F is equal to the ∠ A F C.

In the △ C B F, the side B C is equal to the side B F; therefore the ∠ B C F is equal to the ∠ B F C.

If the equal ∠s B C F and B F C be added to the equal ∠s A C F and A F C (as in Fig. 1), or taken from them (as in Fig. 2), the sums or differences A C B and A F B will be equal.

(If the line C F passes through the point B, it follows immediately that the ∠s A C B and A F B (*i. e.*, D F E) are equal, as in that case these are ∠s at the base of the isosceles △ C A F.)

Therefore the △s A C B and A F B have two sides (A C and C B), and the included ∠ (A C B) of the one equal to two sides (A F and F B), and the included ∠ (A F B) of the other, each to each.

Therefore these △s are also equal to each other in all other respects.

But the △ A F B is the same as the △ D F E.

It has, then, been proved that the △s A C B and D F E are equal to each other in every respect.

> **If the eighth proposition be proved in this manner, the study of the difficult seventh proposition may be postponed till a later period.**

PROPOSITION IX.

To bisect a given rectilineal angle — that is, to divide it into two equal angles.

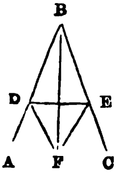
> **For the construction employed in this proposition we require to be able,—**
>
> **1. To join two given points by a straight line.**

2. From the greater of two given lines to cut off a part equal to the less. (Prop. III.)

3. On a given right line to construct an equilateral △. (Prop. I.)

To demonstrate that the construction effects what is required, we must know—

That if two △s have the three sides of the one equal to the three sides of the other, each to each, their ∠s also are equal. (Prop. VIII.)



Suppose A B C to be the given ∠.

In the line B A take any point—as D.

From the line B C cut off a part, B E, equal to B D. (Prop. III.)

Join the points D and E by the right line D E.

On the line D E, and on the side remote from B, construct an equilateral △, D E F. (Prop. I.)

Join the points B and F by the right line B F. Then the right line B F divides the given ∠ into two equal parts.

This is shown as follows:—

D B F and E B F are two △s, in which the sides D B and B E are equal (having been made so in the construction); the sides D F and F E are equal (D E F being an equilateral △); and the side B F is common to the two △s: so that the three sides of the one △ are equal to the three sides of the other, each to each.

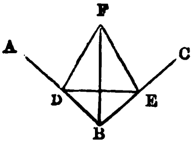
It was proved in the eighth proposition, that in two such △s the ∠s are equal, each to each.

Therefore, in the △s D B F and E B F, the ∠s D B F and E B F are equal.

But the ∠s D B F and E B F are the parts into which the ∠ D B E is divided by the line B F.

It has been proved, therefore, that the line B F bisects* the ∠ D B E.

Let the proposition next be gone through with an ∠ of a different size and in a different position, as in the annexed figure.



PROPOSITION X.

To bisect a given finite straight line, that is, to divide it into two equal parts.

For the construction in this proposition we must be able,—

1. On a given finite right line to construct an equilateral △. (Prop. I.)
2. To bisect a given rectilineal ∠.

To prove that the construction accomplishes what is required we must know that—

If two △s have two sides of the one equal to two sides of the other, each to each, and have also the ∠s included between these pairs of sides equal, then the △s are also equal in every other respect. (Prop. IV.)

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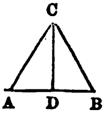
* It is obvious that by successive bisections an angle may be divided into 4, 8, 16, or (generally) 2^n equal parts. It is impossible by ordinary geometry to divide an ∠ into three equal parts, or into any number of equal parts which is a multiple or power of three, with the exception of a right ∠. By the aid of the 32nd proposition of Book I., as applied to an equilateral △, we can obtain the third part of two right ∠s, and by bisecting that, the third part of one right ∠. Certain propositions in the Fourth Book enable us to find the fifth, and the fifteenth part of a right ∠.

Suppose A B to be the given finite right line.



On A B describe an equilateral* △ A B C. (Prop. I.)

From the point C draw a line bisecting the ∠ A C B. (Prop. IX.)

Let this line meet the line A B in the point D.

The line A B is now bisected at the point D.

For in the △s A C D and B C D the sides A C and B C are equal, being sides of an equilateral △. The side C D is common to the two △s, and the ∠ A C D is equal to the ∠ B C D,—the whole ∠ A C B having been divided into two equal parts by the line C D. Thus two sides and the included ∠ of the one △ are equal respectively to two sides and the included ∠ of the other △.

These △s, therefore, are equal in every other respect. (Prop. IV.)

Among these respects is that the side A D is equal to the side B D.

The line A B, therefore, has been divided into two equal parts at the point D.

> Let the proposition now be gone through with the equilateral △ constructed on the other side of A B.

Euclid seems to have supposed that it is a simpler process to bisect an ∠, than to bisect a straight line, and that the latter process is dependent on the former. In this he is wrong. If we go through the process which is *assumed* to be gone through in bisecting the ∠ A C B in the above figure, we shall find that we bisect the line A B, without taking any account of the ∠ A C B. Thus:—

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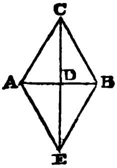
* It is enough for the purpose to describe an *isosceles* △ on A B, with sides of any length. But the construction of an equilateral △ is simpler than that of any other isosceles △.

On A B describe the equilateral △ A C B.

On A B also describe the equilateral △ A E B, on the side remote from C.

Join the points C and E by the line C E, intersecting A B in the point D.

A B is bisected at the point D.



For if we suppose the △ C B E to be applied to the △ C A E in such a manner that, C E remaining common to the two △s, C B and B E may lie on the same side of E C as C A and A E, it will follow (from the seventh proposition) that the point B will coincide with the point A. Therefore the line B D will coincide in direction and length with A D, and must, therefore, be equal to it. In other words, A B has been divided at D into two equal parts.

PROPOSITION XI.

To draw a straight line at right angles* to a given straight line from a given point in the same.

To make the construction we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. From the greater of two given straight lines to cut off a part equal to the less. (Prop. III.)
3. On a given straight line to construct an equilateral △. (Prop. I.)

To prove that the construction is correct we must know that—

If two △s have the three sides of the one equal to the three sides of the other, each to each, then the ∠s of the △s are equal. (Prop. VIII.)

Let A B be the given right line, and C the given point in it. (The line A B may be of any length, and may be produced either way if necessary.)

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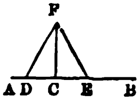
* That is, so that the adjacent ∠s which it makes with the givon line may be equal to each other. (Def. XI.)



In the line C A take any point—as D. From the line C B cut off a part C E, equal to C D. (Prop. III.)

On the line D E construct an equilateral △, D F E.[*]

Join the points F and C by the right line F C, which will be the line required.

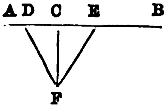
Proof. In the △s D F C and E F C the sides D F and F E are equal, being sides of an equilateral △. The sides D C and E C are equal, C E having been cut off of the same length as C D. The side F C is common to both. Thus the three sides of the one △ are equal to the three sides of the other, each to each.

Consequently the ∠s of the two △s are equal. (Prop. VIII.)

Among these, the ∠s F C D and F C E are equal.

Therefore F C stands upon the line A B in such a manner as to make the adjacent ∠s equal. In other words, it is *perpendicular* to A B. (See Def. XI.)

Let the proposition next be gone through with the figure in the following position:—



There is usually appended to this proposition a corollary, to the effect that by the aid of it it may be proved that two straight lines cannot have a common segment, that is, cannot coincide for a certain distance and then diverge from one another. This ought to be set down as an

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[*] See the note on the last proposition.

axiom. It is assumed and made use of long before we come to the eleventh proposition. In the fourth proposition, for example, where certain lines are placed on one another, *i.e.*, so as to coincide in direction, it is necessary that we should assume that, if they coincide for any portion of their length, they cannot afterwards diverge. It is assumed again in the eighth. Moreover, the proof attempted in the corollary is insufficient. It assumes that all right ∠s are equal; a proposition for the proof of which the truth of this very axiom is assumed (see p. 12), and which could not itself be laid down as an axiom (as is often done) without the very same assumption. If it were regarded as yet unproved that two right lines cannot have a common segment, there would be no ground for assuming that the two equal ∠s formed when one line is perpendicular to another must always be of the same size.

PROPOSITION XII.

To draw a right line perpendicular, or at right angles to a given right line of indefinite length, from a given point without it.*

For the construction in this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. To describe a circle with a given centre, and at a given distance from that centre. (Post. III.)

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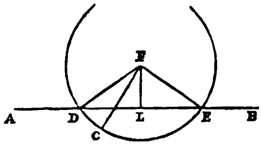
* That is, so that the adjacent ∠s which it forms with the given line may be equal to each other. (Def. XI.)

3. To bisect a given rectilineal ∠. (Prop. IX.)

To prove that the construction accomplishes what is required, we must know that,—

1. Any two radii of the same circle are equal. (Def. XIV.)
2. If two △s have two sides of the one equal to two sides of the other, each to each, and have the ∠s between the said pair of sides in the one △ equal to the ∠ between the said pair of sides in the second △, then those △s will also be equal in every other respect. (Prop. IV.)



Let A B be the given line.

Let F be the given point without it.

Take a point C on the other side of the line A B.

Join F and C by the right line F C.

With F as centre, and at the distance F C, describe a circle, cutting the line A B in the points D and E.

Join F and D, and F and E, by the right lines F D and F E.

From the point F draw the right line F L, bisecting the ∠ D F E, and meeting the line D E in the point L. (Prop. IX.)

L is the line required.

Proof. In the △s D F L and E F L the sides D F and F E are equal, because they are radii of the same circle; the line F L is common to the two △s; and the ∠ D F L is equal to the ∠ E F L, because they are the two equal ∠s into which the ∠ D F E is divided by the line F L. That is to say, the two △s have two sides of the one equal to two sides of the other, each to each, and the ∠ between the said pair of sides in the first △ equal to the ∠ between the said pair of sides in the second △.

Consequently, as was proved in the fourth proposition, these two △s are also equal in every other respect.

Among these respects is that the ∠ F L D is equal to the ∠ F L E.

The ∠s F L D and F L E are adjacent ∠s.

It has thus been shown that the line F L, meeting the line A B, makes the adjacent ∠s equal to one another. In other words, it is perpendicular to the line A B. (Def. XI.)

PROPOSITION XIII.

The angles which one right line, meeting another, forms with it on the same side, are either two right angles, or together equal to two right angles.

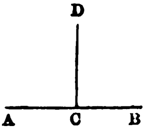
> To prove this proposition we must be able to draw a perpendicular to a given straight line from a given point in the same. (Prop. XI.)
>
> We must also know,—
>
> 1. That magnitudes which are equal to the same are equal to each other.
> 2. That if the same quantity be added to each of two equal quantities, the sums will be equal. (Ax. II.)

Suppose the straight line D C to meet the straight line A B.

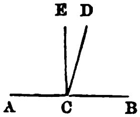
The ∠s which D C makes with A B must be either equal or unequal.

If they are equal, they are called right ∠s.

But suppose they are unequal, then it is easily shown that their sum is the same as that of two right ∠s.



For, from the point C draw a straight line C E perpendicular to A B—that is, making the adjacent ∠s E C A and E C B equal to one another.



It is evident that the ∠ E C B is the sum of the two ∠s E C D and D C B.

Therefore, the sum of the two ∠s A C E and E C B is the same as that of the three ∠s A C E, E C D, and D C B.

Again the ∠ A C D is the sum of the ∠s A C E and E C B.

Therefore, the sum of the two ∠s A C D and D C B is the same as the sum of the three ∠s A C E, E C D, and D C B.

The two right ∠s A C E and E C B were before shown to be equal to the sum of the same three ∠s.

Therefore, the sum of the ∠s A C D and D C B is the same as that of the right ∠s A C E and E C B; and as all right ∠s are equal (see p. 13), the sum of the ∠s A C D and D C B is the same as that of any two right ∠s.

PROPOSITION XIV.

If two straight lines meet a third at the same point, but on opposite sides of the line, and make the adjacent angles together equal to two right angles, then those two straight lines are in one and the same straight line.

In other words, we have to prove that if either of the straight lines be produced, the produced part will coincide in direction with the other of the two.

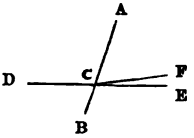
For the proof of this proposition, we must know that,—

1. If one straight line meets another, the adjacent ∠s which it forms with it are together equal to two right ∠s. (Prop. XIII.)
2. Quantities that are equal to the same are equal to each other. (Ax. I.)
3. If the same quantity be taken away from each of two equal quantities the remainders are equal. (Ax. III.)

The proof of this proposition is of the indirect kind, and consists in showing that it leads to an absurdity to suppose that, if one of the two straight lines be produced, the produced part does not coincide in direction with the other line.

Let D C and E C be two straight lines, meeting the line A B at the point C, but on opposite sides of the line A B, and making the ∠s D C A and A C E together equal to two right ∠s.

We have to show that if one of the lines (as D C) be produced, the produced part coincides in direction with C E.



This is shown as follows:—

Suppose it possible that the line D C produced did not coincide in direction with C E, but lay in some other direction, as C F.

We should then have the line A C meeting the straight line D F at the point C.

The adjacent ∠s D C A and A C F, so formed, would together be equal to two right ∠s, as was proved in the last proposition.

Now we know, to start with, that the ∠s A C D and A C E are together equal to two right ∠s.

We should be compelled to admit, therefore, that the ∠s D C A and A C F taken together are equal to the ∠s D C A and A C E taken together.

If this were so, it would follow that, on taking the

∠ D C A away from both these equal sums, the remainders, A C E and A C F are equal.

But this is impossible, for the ∠ A C F is only a part of the ∠ A C E.

This absurdity is a necessary inference from the supposition that the line D C, when produced, does not coincide in direction with C E.

But a supposition which leads of necessity to an impossibility is itself absurd.

Consequently, it is impossible that the line D C when produced should *not* coincide in direction with C E. Therefore, as D C produced must lie in some direction or other, it must coincide with C E.

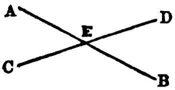
PROPOSITION XV.

If two straight lines cut one another, the vertically opposite angles will be equal to one another.

To prove this proposition we must know,—

1. That if one straight line meets another, the adjacent ∠s which it forms with it are together equal to two right ∠s.
2. If the same quantity be taken from two equal quantities, the remainders are equal. (Ax. III.)

Let A B and C D be two straight lines intersecting in the point E. We have to prove that the ∠ A E D is equal to the ∠ C E B, and the ∠ A E C to the ∠ D E B.



Proof. A E is a straight line meeting the line C D in the point E, and forming adjacent ∠s A E C and A E D.

Consequently, these two ∠s are together equal to two right ∠s.

Again, C E is a straight line, meeting the straight line A B in the point E, and forming adjacent ∠s A E C and C E B.

Consequently, the ∠s A E C and C E B are together equal to two right ∠s. (Prop. XIII.)

But quantities that are equal to the same are equal to each other.

Therefore, the sum of the ∠s A E C and A E D is equal to the sum of the ∠s A E C and C E B, each sum being equal to the sum of two right ∠s.

Take away the ∠ A E C from each of these equal sums, and the remainders will be equal, *i.e.*, the ∠ A E D is equal to the ∠ C E B.

Again, since the straight line D E meets the line A B, the adjacent ∠s A E D and D E B are together equal to two right ∠s, and it was before shown that the ∠s A E D and A E C are together equal to two right ∠s.

Consequently, the sum of the ∠s A E D and D E B is equal to the sum of the ∠s A E D and A E C.

Take away the ∠ A E D from each of these equal sums, and the remainders will be equal, *i.e.*, the ∠ A E C is equal to the ∠ D E B.

PROPOSITION XVI.

If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite* angles.

For the construction employed in this proposition we must be able,—

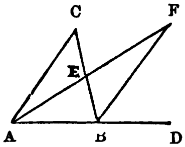
---

* That is, not adjacent to the exterior angle.

1. To join two given points by a right line. (Post. I.)
2. To produce a given finite right line to any length. (Post. II.)
3. From the greater of two given lines to cut off a part equal to the smaller. (Prop. III.)
4. To bisect a given finite straight line. (Prop. X.)

For the demonstration of the truth of the proposition by the aid of the construction we must know,—

1. That if two straight lines cut one another, the vertically opposite ∠s are equal. (Prop. XV.)
2. That if two △s have two sides of the one equal to two sides of the other, each to each, and also the ∠ included by the said pair of sides in the first △ equal to the ∠ included by the corresponding pair of sides in the second △, then the △s will also be equal in every other respect. (Prop. IV.)



Let A B C be a △, of which one side, A B, has been produced. We have to show that each of the ∠s B C A and B A C (which are *not* adjacent to the ∠ C B D), is less than the exterior ∠ C B D.

We must first take the ∠ A C B, which is *opposite* to the side A B which has been produced.

In order to compare it with the ∠ C B D, make the following construction:—

Bisect the line C B in the point E.

Join the points A and E by the right line A E. (Post. I.)

Produce the line A E to any length. (Post. II.)

From the produced part of A E cut off a part, E F, equal to A E. (Prop. III.)

Join the points F and B by the right line F B.

We thus get two △s B E F and C E A, in which the sides B E and C E are equal (being the equal parts into which the line B C has been divided); A E and E F are equal (E F having been cut off of the same length as A E), and the ∠ B E F is equal to the ∠ C E A, because they are vertically opposite ∠s, formed by the intersection of the lines B C and A F. So that the two △s B E F and C E A have two sides and the included ∠ of the one equal respectively to two sides and the included ∠ of the other.

These two △s, therefore, are also equal to each other in every other respect. (Prop. IV.)

Among these respects is that the ∠ E C A is equal to the ∠ E B F.

But the ∠ E B F is less than the ∠ C B D. Therefore, the ∠ E C A, which is equal to the ∠ E B F, is also less than the ∠ C B D.

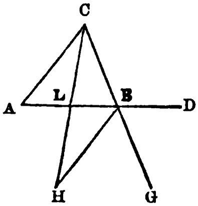
It has thus been shown that the exterior ∠ C B D is greater than that one of the interior ∠s which is opposite to the side produced.

It still remains to be shown that the ∠ C A B is also less than the ∠ C B D.

The ∠ C A B cannot be compared directly with the ∠ C B D; but, by producing the line C B to any point G, we get an ∠ A B G equal to the ∠ C B D (being vertically opposite to it), with which the ∠ C A B may be compared in the same way as that in which the ∠ A C B was compared with the ∠ C B D.

For this purpose bisect the line A B in the point L. Join the points C and L by the right line C L. Produce the line C L to any length. From the produced part cut off a part, L H, equal to the line C L. Join the points H and B by the right line H B.

We thus get two △s, A L C and B L H, in which the side A L is equal to the side B L (A B being bisected in the point L); the side C L is equal to the side L H (L H having been cut off of the same length as C L); and the ∠s A L C and B L H are equal, being vertically opposite. That is to say, two sides and the ∠ between them in the △ A L C are equal respectively to two sides and the ∠ between them in the △ B L H.



It follows, therefore (Prop. IV.), that the △s A L C and B L H are also equal in every other respect.

Among these respects is that the ∠ C A L is equal to the ∠ L B H.

But the ∠ L B H is less than the ∠ A B G (being a part of it).

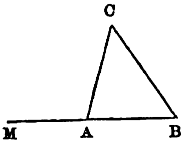
Therefore the ∠ C A L is also less than the ∠ A B G.

But the ∠ A B G is equal to the vertically opposite ∠ C B D.

Therefore the ∠ C A L, being less than the ∠ A B G, is also less than the ∠ C B D.

It has thus been shown that the ∠s A C B and C A B are each less than the exterior ∠ C B D, formed by producing the side A B.

Let this proposition now be demonstrated with the following variations of the figure :—

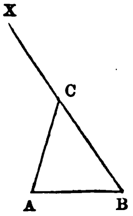


1. Produce the side B A to any point M, and prove that the ∠ A C B is less than the ∠ C A M. The following directions will suffice : — 1. Bisect the line A C in the point P. 2. Join B and P by a straight line. 3. Produce the line B P and cut off a part, P Q, equal to P B. 4. Join the points Q and A by

a right line. Then show that the △ C P B is equal in all respects to the △ P Q A. And so, that the ∠ A C B is less than the ∠ C A M.

Then producing C A to N, show that the ∠ C B A is less than the ∠ B A N, and therefore less than the ∠ C A M.

2. Produce the side B C to any point X, and show that the ∠s C A B and C B A are each of them less than the ∠ A C X.



The rule for proceeding is always the following:—When a side of the △ has been produced, the ∠ that is to be compared with the exterior ∠ is that ∠ of the △ which is *opposite the side produced.* For making the necessary construction, bisect that side of the △ which is one of the sides containing the exterior ∠. Join the bisecting point with the opposite vertex of the △. Produce this line through the point of bisection, and cut off a part equal to the part within the △. Lastly, connect the point so obtained, by a right line, with the vertex of the exterior ∠.

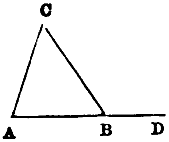
PROPOSITION XVII.

Any two angles of a triangle are less than two right angles.

For the proof of this proposition we must be able to produce a given straight line to any length, and we must know,—

1. That if one side of a △ be produced, the exterior ∠ is greater than either of the two interior and opposite ∠s. (Prop. XVI.)
2. That if one straight line meets another, the adjacent ∠s which it forms with it are together equal to two right ∠s. (Prop. XIII.)
3. That if the same quantity be added to each of two unequal quantities, the sum of that quantity

and the larger of the two unequals is greater than the sum of that quantity and the smaller of the unequals. (Ax. IV.)



Let A B C be a △.

Take any two of its ∠s, say C A B and C B A.

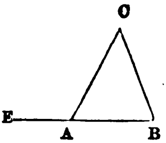
Produce one of the sides of the △, so as to form an exterior ∠ that shall be adjacent to one or other of the two ∠s C A B and C B A. As, for example, produce the side A B to the point D.

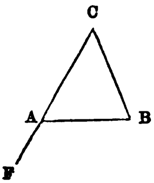
It follows from the last proposition that the ∠ CBD is greater than the ∠ C A B. If the ∠ C B A be added to each of these unequal ∠s, the sum of the ∠s C B D and C B A will be greater than the sum of the ∠s C A B and C B A.

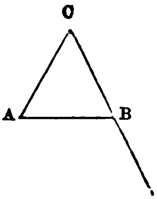
But the ∠s C B D and C B A are adjacent ∠s, formed by the straight line C B meeting the straight line A D, and are therefore equal to two right ∠s.

Consequently, the ∠s C A B and C B A are together less than two right ∠s.

The proposition might have been proved equally well by either of the following constructions:—







In each case we should have an exterior ∠ greater than one or other of the two given ∠s of the △; enabling us to show that the sum of the two given ∠s, being less than that of one of them and the exterior ∠, is less then two right ∠s.

Whichever two ∠s of the △ are selected, it does not matter which side of the △ is produced, provided it is produced in a direction *passing through the vertex of one of the two ∠s fixed upon.*

PROPOSITION XVIII.

If one side of a triangle be greater than another, the angle opposite the greater side is greater than the angle opposite the less.*

For the construction requisite in this proposition, we must be able,—

1. From the greater of two straight lines to cut off a part equal to the less. (Prop. III.)
2. To join two given points by a straight line. (Post. I.)

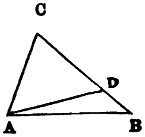
To prove the proposition by the aid of the construction, we must know that,—

1. If two sides of a △ are equal, the ∠s opposite to those sides are also equal. (Prop. V.)
2. If one side of a △ be produced, the exterior ∠ is greater than either of the two interior opposite ∠s. (Prop. XVI.)

Let A B C be a △ in which the side C B is longer than the side C A.

We have to show that the ∠ C A B is larger than the ∠ C B A.

From the greater side, C B, cut off a part, C D, equal to the less.



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* The enunciation of this proposition, as it stands in the common editions of Euclid, is almost sure to mislead the beginner. It runs :—"The greater side of every △ is opposite the greater ∠." This ambiguous statement is always to be interpreted in the form given above.

Join the points A and D by the right line A D.

The △ A C D, which is thus formed, is an isosceles △. Therefore, the ∠ C A D is equal to the ∠ C D A. (Prop. V.)

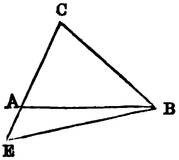
The ∠ C D A is the exterior ∠ formed by the production of the side B D in the △ A B D. Therefore, the exterior ∠ C D A is greater than the interior and opposite ∠ D B A. (Prop. XVI.)

Now the ∠ C A B is greater than the ∠ C A D.

It is therefore also greater than the ∠ C D A, which is equal to the ∠ C A D.

Much more then must the ∠ C A B be greater than the ∠ C B A, which was before shown to be less than the ∠C D A.

This proposition may also be proved in the following manner :—



Produce C A, the shorter of the two sides, and cut off from the produced line a part, C E, equal to the line C B.

Join the points E and B.

The △ E C B, which is thus formed, is an isosceles △, consequently the ∠ C E B is equal to the ∠ C B E.

Now, the ∠ CAB is the exterior ∠ formed by producing the side E A in the △ B E A. Consequently, the ∠ C A B is greater than the interior and opposite ∠ A E B.

It follows, therefore, that the ∠ C A B is also greater than the ∠ C B E, which is equal to the ∠ C E B.

Much more is the ∠C A B greater than the ∠C B A, which is less than the ∠ CB E.

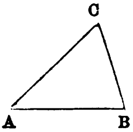
PROPOSITION XIX.

If one angle of a triangle be greater than another, the side which is opposite the greater angle is greater than the side which is opposite to the less.*

The proof of this proposition is of the indirect kind

For the proof of it we must know,—

1. That if two sides of a △ are equal, the ∠s opposite the equal sides are also equal. (Prop. V.)
2. That if one side of a △ is greater than another, the ∠ opposite the greater side is greater than the ∠ opposite the less. (Prop. XVIII.)



Let A B C be a △, with respect to which we know that the ∠ C B A is greater than the ∠ C A B. It has to be proved that the side C A is longer than the side C B.†

Three alternatives only are possible, one of which must be true. Either the side

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* This proposition is often expressed thus:—"The greater ∠ of every △ is subtended by the greater side, or has the greater side opposite to it." The meaning of this is precisely the same as what is given above.

† Let the beginner carefully distinguish between this proposition and the last. In the 18th proposition we were supposed to know something about the length of two sides of a △, but nothing about the ∠s, except what we could *deduce* from our knowledge respecting the sides. In the 19th proposition we are supposed to know something respecting two ∠s of a △, but nothing respecting the sides, except what we can deduce from our knowledge respecting the ∠s. Compare the 5th and 6th propositions.

C B is greater than the side C A, or C B is equal to C A, or C B is less than C A.

Now, it is impossible that C B should be equal to C A, for, if it were, the ∠ C A B would be equal to the ∠ C B A; whereas, we know that the ∠ C B A is greater than the ∠ C A B.

It is also impossible that C A should be less than C B, for, if it were, the ∠ C B A would be less than the ∠ C A B; whereas, we know that the ∠ C B A is greater than the ∠ C A B. It follows, therefore, that the side C A is greater than the side C B.

PROPOSITION XX.

Any two sides of a triangle are together greater than the third side.*

For the construction employed in this proposition, we must be able,—

1. To produce a given line to any length. (Post. II.)
2. From the greater of two given lines to cut off a part equal to the less. (Prop. III.)
3. To join two given points by a straight line. (Post. I.)

For the proof of the proposition we must know,—

1. That if equal quantities are added to the same quantity, the sums are equal.
2. That if two sides of a △ are equal, the ∠s opposite those sides are also equal. (Prop. V.)

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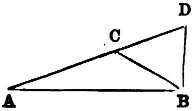
* Let the beginner remember that the object of this proposition is not to *convince* him of the truth stated, but to show how it may be connected with and deduced from the fundamental axioms and definitions.

3. That if one ∠ of a △ be greater than another, the side opposite the greater ∠ is greater than the side opposite the less. (Prop. XIX.)



Let it be required to prove that the sides A C and C B in the △ A C B are together greater than the side A B.

Produce one of the two sides whose sum is to be compared with the third, as A C, to any length, and from the produced part cut off a part, C D, equal to C B.

Join the points D and B by the right line D B.

In the △ D C B the sides D C and C B are equal. It follows, therefore (according to the fifth proposition), that the ∠ C D B is equal to the ∠ C B D.

Now, the ∠ A B D is greater than the ∠ C B D.

Therefore the ∠ A B D is also greater than the ∠ C D B, which is equal to the ∠ C B D.

Consequently, in the △ A B D, one ∠, namely A B D, is greater than another, namely A D B.

Therefore (as was proved in the nineteenth proposition), the side A D is greater than the side A B.

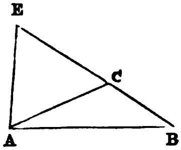
Now, A D is made up of the parts A C and C D, of which C D is equal to C B.

Therefore, the sum of A C and C D (*i.e.*, the line A D) is equal to the sum of A C and C B. (Ax. II.)

The line A D was shown to be greater than the line A B.

Therefore, the sum of the lines A C and C B is also greater than the line A B.

It would have done equally well if the side B C had been produced, and a part, C E, cut off equal to the side C A. We should then have had the ∠ B A E greater than the ∠ C A E, and, consequently, greater than the ∠ C E A, which is equal to C A E. It would have followed that the side B E in the △ A B E is longer than the side B A. Then, B C and C E, taken together, being equal to B C and C A taken together, it would have followed that the sum of B C and C A is greater than the line B A.



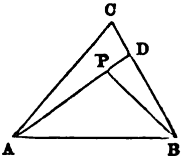
PROPOSITION XXI.

If from the extremities of one side of a triangle lines be drawn to a point within the triangle, these two lines will together be less than the other two sides of the triangle, but will contain a greater angle.

For the proof of this proposition we must know that,—

1. Any two sides of a △ are together greater than the third side. (Prop. XX.)
2. If the same quantity be added to each of two unequal quantities, the sum of that quantity, and the greater of the two unequals, will be greater than the sum of the same quantity, and the lesser of the unequals. (Ax. IV.)
3. The exterior ∠, formed by producing a side of a △, is greater than either of the two interior opposite ∠s. (Prop. XVI.)

Let A B C be a △; and suppose that lines A P and B P are drawn from the extremities of the side A B, to the point P, taken within the △. It has to be proved that the sum of the lines A P and P B is less than the sum of the sides A C and C B, and that the ∠ A P B is greater than the ∠A C B.



To prove this, produce the line A P, to meet the side C B in the point D.

1. In the △ A C D, the sum of the sides A C and C D is greater than the side A D.

Let the line D B be added to each of these unequals, —that is, to the sum of A C and C D, and to A D.

It will follow that the sum of A C, C D, and D B, will be greater than the sum of A D and D B. But the sum of A C, C D, and D B is the same as the sum of A C and C B. Therefore, the sum of the lines A C and C B is greater than the sum of the lines A D and D B.

Again, in the △ P D B, the sum of the sides B D and D P is greater than the side B P.

Let the line P A be added to each of these unequals, —that is, to the sum of B D and D P and to B P.

It will follow that the sum of B D, D P, and P A, will be greater than the sum of P B and P A.

But the sum of B D, D P, and P A, is the same as the sum of B D and D A.

Therefore, the sum of B D and D A is greater than the sum of B P and P A.

Now it was before shown that the sum of A C and C B is greater than the sum of A D and D B.

Much more, therefore, is the sum of A C and C B greater than the sum of A P and P B, which is less than the sum of A D and D B.

2. On looking at the △ B D P, with its produced side D P A, it will be seen that the ∠ A P B is an exterior ∠ of that △, and the ∠ P D B one of the interior opposite ∠s.

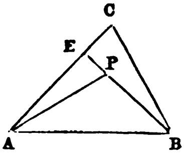
Consequently (Prop. XVI.), the ∠ A P B is greater than the ∠ P D B.

Again, the ∠ P D B is an exterior ∠ of the △ A C D, formed by the production of the side C D.

The ∠ P D B, therefore, is greater than the ∠ A C D.

But it was before shown that the ∠ A P B is greater than the ∠ P D B. Much more, therefore, must the ∠ A P B be greater than the ∠ A C B, which is less than the ∠ P D B.

In proving this proposition, it would have done equally well if the line B P had been produced to meet the side A C in E.



It would then have been shown that the sum of A C and C B is greater than that of A E and E B; and the sum of A E and E B greater than that of A P and P B. From which it would have been inferred that the sum of A C and C B is greater than that of A P and P B.

Secondly, it would have been shown that the exterior ∠ A P B is greater than the interior opposite ∠ A E P; and that the ∠ A E P is, in its turn, an exterior ∠ of the △ E C B, and, therefore, greater than the interior and opposite ∠, E C B. From which it would have been inferred, that the ∠ A P B is greater than the ∠ A C B.

PROPOSITION XXII.

To make a triangle, the sides of which shall be equal respectively to three given straight lines, any two of which are greater than the third.

For the construction in this proposition we must be able,—

1. To describe a circle with a given centre, at a given distance from that centre. (Post. III.)

2. To join two given points by a straight line. (Post I.)
3. From a given point to draw a straight line equal in length to a given finite straight line. (I. 2.)
4. From a given line to cut off a part equal in length to a given finite right line. (Prop. III.)

To prove that the construction effects what is required we must know,—

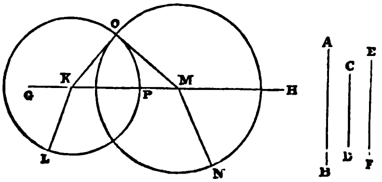
1. That all radii of the same circle are equal in length. (Def. XV.)
2. That magnitudes which are equal to the same are equal to one another. (Ax. I.)

Let AB, CD, and EF be the three given lines—any two of which are greater than the third.

Take a line GH, of unlimited length.

In this line take any point, K.

From the line KH cut off a part, KM, equal to AB. (Prop. III.)



From the point K draw a line equal in length to the line CD. Let it be the line KL. (Prop. II.)

From the point M draw a line, MN, equal in length to the line EF.

With K as centre, and at the distance KL, describe a circle. (Post. III.)

With M as centre, and at the distance MN, describe another circle.

These two circles cannot lie outside each other, but must be partly within and partly without each other. For (P being the point where the circle LPO intersects the line GH) KP is equal to KL, and is therefore equal to C D. Now M P must be less than E F, and therefore less than a radius of the circle described with the centre M, because C D and E F together are greater than A B, and therefore greater than K M. Consequently, K P and a radius of the circle described with the centre M must be greater than K M; *i.e.*, M P is less than a radius of the circle described with the centre M.

It follows, therefore, that a part of each circle lies within the other. Consequently, they cut each other in two points. (See Ax. C.)

Take O, one of these points of intersection, and join K and O by the right line K O, and M and O by the right line M O.

The △ K O M will be the △ required.

For the side K O is equal to the line K L, both being radii of the same circle.

But K L was drawn equal to C D.

Therefore also K O is equal to C D.

Again, the side M O is equal to M N, both being radii of the same circle.

But M N was drawn equal to E F. Therefore, M O is equal to E F.

Lastly, the side K M was cut off of the same length as A B.

It has thus been shown, that the three sides of the △ K O M are equal respectively to the three given straight lines A B, C D, and E F.

PROPOSITION XXIII.

From a given point in a straight line to draw a straight line making with the given straight line an angle equal to a given angle.

For the construction employed in this proposition we must be able,—

1. To join two points by a straight line. (Post. I.)
2. From a given point to draw a straight line equal to a given straight line. (Prop. II.)
3. To describe a circle with a given centre and at a given distance from that centre. (Post. III.)
4. From the greater of two given straight lines to cut off a part equal to the less. (Prop. III.)

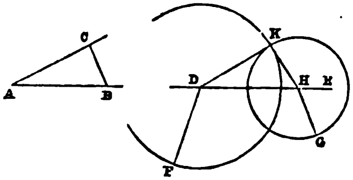
To prove that the construction effects what is required we must know,—

1. That radii of the same circle are equal. (Def. XIV.)
2. That magnitudes which are equal to the same are equal to one another. (Ax. I.)
3. That two sides of a △ are together greater than the third. (Prop. XX.)
4. That if two △s have the three sides of the one equal respectively to the three sides of the other, the △s will also be equal in every other respect. (Prop. VIII.)

Let B A C be the given rectilineal ∠.

Let D E be the given straight line, and D the given point in it.

It is required to draw from the point D a straight line, which shall make with the line D H an ∠ equal to the ∠ B A C.



In the line A C take any point, C.

In the line A B take any point, B.

Join the points C and B by the straight line C B.

From the point D draw a line, D F, equal in length to the line A C. (Prop. II.)

From the line D E cut off a part, D H, equal in length to the line A B. (Prop. III.)

From the point H draw a line, H G, equal in length to the line B C. (Prop. II.)

With D as centre and at the distance D F describe a circle. (Post. III.)

With H as centre and H G as radius describe another circle.

Because A C and C B are together greater than A B (any two sides of a △ being greater than the third side), and A C, C B, and A B are equal respectively to D F, H G, and D H, it follows that D F and H G are together greater than D H.

Consequently the two circles described with the centres D and H will intersect one another.

Let K be one of the points where they intersect.

Join D and K by the straight line D K.

D K will be the line required.

For, join K and H by the right line K H.

In the △ D K H the side D K is equal to the line D F, both being radii of the same circle.

But D F is equal to A C.

Therefore D K is equal to A C. (Ax. I.)

The side K H is equal to the line H G.

But H G is equal to C B.

Therefore K H is equal to C B.

And lastly, the side D H was cut off of the same length as A B.

Thus, in the △s B A C and H D K, the sides of the one are equal respectively to the sides of the other.

Consequently the △s are also equal in every other respect.

Among these respects is that the ∠ K D H is equal to the ∠ C A B.

That is to say, the line D K has been drawn so as to make with the line D E an ∠ equal to the given ∠ C A B.

The directions for the construction, as set down in the ordinary editions of Euclid, involve making a triangle, the three sides of which shall be equal respectively to the three sides of another triangle, and, moreover, so that the sides shall lie in a given position. Now the 22nd proposition does not provide for this, and the construction as given above simply states explicitly what is necessary to effect this.

It will be convenient, before entering upon the 24th proposition, to establish the following introductory proposition, of which use is made in the course of the 24th proposition.

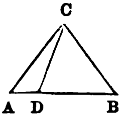
If a straight line be drawn from the vertex of a triangle to any point in the base between its extremities, that line will be shorter than one or other of the other two sides of the triangle.*

This proposition admits of two cases :—

1. Where the △ is an isosceles △.

2. Where one of the two sides is longer than the other.

1. Let A C B be an isosceles △, and C D a line drawn from the vertex to the point D in the base.



It has to be shown that the line C D is less than either of the sides A C, C B.

The ∠ C D A is the exterior ∠ of the △ C B D, formed by the production of the side B D.

Therefore, the ∠ C D A is greater than the ∠ C B D.

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* It *may* be less than either—but *must* be less than one or other.

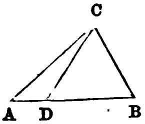
But since A C B is an isosceles △, the ∠ C A B is equal to the ∠ C B A. (Prop. V.)

Consequently the ∠ C D A, which is greater than the ∠ C B A, is also greater than the ∠ C A B.

But since, in the △ C D A, the ∠ C D A is greater than the ∠ C A D, it follows (according to the 19th proposition) that the side C A is longer than the side C D. And since C B is equal to C A, C B is also longer than C D.

2. Let A C B be a △ in which the side A C is longer than the side C B.

It has to be shown that the line C D is shorter than the longer of the two sides.



Because C A is longer than C B, it follows (according to the 18th proposition) that the ∠ C B A is greater the ∠ than C A B.

C D A is the exterior ∠ of the △ C B D formed by the production of the side B D.

Consequently the ∠ C D A is greater than the ∠ C B D.

Much more therefore is the ∠ C D A greater than the ∠ C A D, which is less than the ∠ C B D.

But since, in the △ C D A, the ∠ C D A is greater than the ∠ C A D, it follows that the side C A is longer than the side C D.

In each of these two cases it will be observed, that the side with which the line drawn to the base is compared may be described as *that side which is not the less of the two.* And what has been demonstrated in the two cases may be summed up by saying, that "If a line be drawn from the vertex of a triangle to any point of the base between its extremities, that line is shorter than that one of the other two sides of the triangle which is not the less of the two."

PROPOSITION XXIV.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the said two sides of one of them greater than the angle contained by the said two sides of the other, the base of that triangle which has the greater angle will be greater than the base of the other.

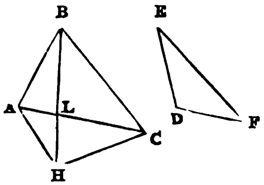
For the construction necessary in this proposition we must be able,—

1. To join two given points by a straight line.
2. From the greater of two given lines to cut off a part equal to the less. (Prop. III.)
3. From a given point in a given straight line to draw a straight line, making with the given straight line an ∠ equal to a given ∠. (Prop. XXIII.)

For proving the proposition by the aid of the construction we must know,—

1. That if a line be drawn from the vertex of a △ to a point in the base, that line is less than that one of the other two sides of the △ which is not the less.
2. That if two △s have two sides of the one equal to two sides of the other, each to each, and have likewise the ∠ included between the said sides in the one △ equal to the ∠ included between the said sides in the other, the △s are also equal in every other respect. (Prop. IV.)
3. That if two sides of a △ are equal, the ∠s opposite those sides are also equal. (Prop. V.)
4. If one ∠ of a △ be greater than another, the side which is opposite the greater ∠ is greater than the side which is opposite the less. (Prop. XIX.)

Let ABC and DEF be two △s, in which the



sides A B and B C are equal to the sides D E and E F, each to each, but the ∠ A B C is greater than the ∠ D E F.

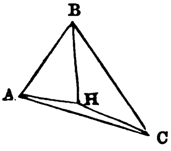
It has to be proved that the base A C is greater than the base D F.

Of the two sides A B, B C, let A B be that which is not the greater of the two (*i.e.*, if the △s are *not* isosceles, let it be the shorter of the two sides).

From the point B draw a line, B L, making with the line A B an ∠ equal to the ∠ D E F, and meeting the base A C in the point L.

B L being a line drawn from the vertex of a △ to a point in the base between its extremities, is shorter than that one of the two sides of the △ A B C which is not the less of the two. That is to say, it is shorter than B C (which we suppose to be either equal to A B, or longer than A B).

Produce B L to any length, and from B L produced cut off a part, B H, equal in length to B C.*



* It is essential for the proof of the proposition that the point H should be on the opposite side of the base A C to the vertex B. For this purpose it is necessary that the line drawn from B, making with one of the sides of the △ A B C an ∠ equal to the ∠ D E F, should make that ∠ with that one of the two sides, A B, B C, which is *not the longer of the two.* If B C be longer than B A, and the line B H be drawn making with B C an ∠ equal to the ∠ D E F, it is quite possible that the extremity, H, of the line cut off equal to B A may fall within the △; in which case the construction does not enable us to establish the proposition.

Join the points A and H and C and H by the right lines A H and C H.

By this construction we get a △, A B H, in which the side A B is (by supposition) equal to the side D E in the △ D E F; the side B H was made of the same length as the side B C or E F; and the ∠ A B H was made equal to the ∠ D E F. That is to say, the two △s A B H and D E F have two sides of the one, with the ∠ between them, equal respectively to two sides of the other with the ∠ between them.

It follows therefore that these two △s are equal in every respect.

Among these respects is that the base A H is equal to the base D F.

It now remains that we should show that the line A H is less than the line A C, which is done by showing that in the △ A H C, the ∠ A H C is greater than the ∠ A C H.

The line B H was made of the same length as the line B C.

Consequently, in the △ H B C, the ∠s B H C and B C H, which are opposite the equal sides, are themselves equal.

Now, the ∠ A H C is greater than the ∠ B H C, and is therefore greater than the ∠ B C H, which is equal to the ∠ B H C.

Moreover, the ∠ A C H is less than the ∠ B C H, and is therefore less than the ∠ A H C, which is greater than the ∠ B C H.

But it was proved in the 19th proposition that if one ∠ of a △ is greater than another, the side which is opposite the greater ∠ is greater than the side which is opposite the less.

Consequently, in the △ A H C, the side A C, which is opposite the larger ∠, A H C, is greater than the side A H, which is opposite the smaller ∠, A C H.

Now it was proved before, that the line A H is equal to the line D F.

Therefore, the line A C, which is greater than A H, is also greater than D F.*

PROPOSITION XXV.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base (or third side) of the one greater than the base (or third side) of the other, the angle contained by the two sides of the one which has the greater base will be greater than the angle contained by the corresponding two sides of the other.

The proof of this proposition is of the indirect kind. No construction is made use of for comparing the ∠s; but it is shown that every supposition respecting the relative size of the two ∠s, except that stated in the enunciation of the

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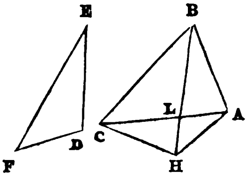
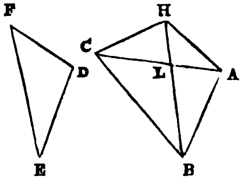
* Let this proposition next be gone through with the following variations in the position of the figures :—





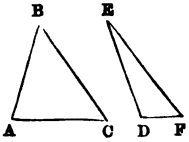
No alteration will be necessary in the wording of the proposition, while the letters are arranged as above.

proposition, must be impossible, because it would oblige us to admit something which is contradicted by what we know to be true. (See p. 28.)

For the proof of the proposition we must know,—

1. That if two △s have two sides of the one equal to two sides of the other, each to each, and have likewise the ∠ included between the said sides of the one equal to the ∠ included between the said sides of the other, then those two △s are also equal in every other respect, and (among those respects) have their bases equal. (Prop. IV.)
2. That if two △s have two sides of the one equal to two sides of the other, each to each, but the ∠ included between the said sides of the one greater than the ∠ included between the said sides of the other, then the base of that △ which has the greater ∠ is greater than the base of that which has the smaller ∠. (Prop. XXIV.)

Let A B C and D E F be two △s in which the sides A B and B C are equal to the sides D E and E F, each to each, and the side A C greater than the side D F. It has to be shown that the ∠ A B C is greater than the ∠ D E F.



Proof. One of the three following alternatives *must* be true, and only one *can* be true.

Either the ∠ A B C is greater than the ∠ D E F, or the ∠ A B C is equal to the ∠ D E F, or the ∠ A B C is less than the ∠ D E F.

Now, the ∠ A B C cannot be *equal* to the ∠ D E F, for, if it were, we should have the two △s, A B C and D E F, with two sides and the included ∠ of the one equal respectively to two sides and the included ∠ of the other. Consequently (according to the fourth proposition), the base A C would be *equal* to the base D F.

But this cannot be the case, for we know, to start with, that the base A C is *greater* than the base D F.

Consequently, the ∠ A B C cannot be equal to the ∠ D E F.

Again, the ∠ A B C cannot be less than the ∠ D E F, for, if it were, then in the two △s, A B C and D E F, we should have the two sides A B and B C equal respectively to the two sides D E and E F, but the ∠ D E F greater than the ∠ A B C. Consequently (according to the 24th proposition), the base D F would be greater than the base A C.

But this is impossible, for we know, to start with, that D F is *not* greater than A C.

It follows, therefore, that since the ∠ A B C cannot be equal to or less than the ∠ D E F, it must be greater than it.

PROPOSITION XXVI.

If two triangles have two angles in the one equal to two angles in the other, each to each, and have likewise one side in the one equal to a side in the other similarly placed with respect to the equal angles, then those triangles will also be equal in every other respect.

For the construction employed in this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. From the greater of two given straight lines to cut off a part equal to the less. (Prop. III.)

For the proof of the proposition we must know,—

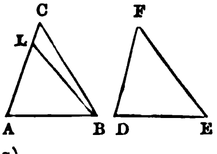
1. That magnitudes which are equal to the same, are equal to one another. (Axiom I.)
2. That if two △s have two sides and the ∠ between them in the one, equal respectively to two sides and the ∠ between them in the other, those △s are also equal in every other respect. (Prop. IV.)
3. That if one side of a triangle be produced, the exterior ∠ is greater than either of the two interior opposite angles. (Prop. XVI.)

This proposition divides itself into two cases:—

1. When the equal sides are opposite neither of the equal ∠s.

2. When the equal sides are opposite respectively to one or other of the pairs of equal ∠s.

The proof of the proposition is (in part) of the indirect kind. It is shown first that we should be compelled to admit an absurdity, if it were supposed possible that one of the other pairs of sides in the △s were unequal, and then, by the application of the fourth proposition, it is proved that the two △s are equal in every other respect.



1. Suppose ABC and DEF to be two △s, with respect to which we know that the ∠ CAB is equal to the ∠ EDE; the ∠ CBA to the ∠ FED, and the side AB equal to the side DE (the sides AB and DE being opposite to *neither* of the pairs of equal ∠s).

It has first to be shown that, besides the sides A B and D E, the △s have also another pair of sides equal.

Take for instance, the sides A C and D F. If these sides were not equal, one would be the greater of the two. Suppose A C were greater than D F.

We should then be able to cut off from A C a part, A L, equal to D F.

Then, if we were to join L and B by the straight line B L, we should have a △ L A B, in which the side L A would be equal to the side D F in the △ D E F, the side A B to the side D E, and the ∠ L A B equal to the ∠ F D E.[*] That is to say, these two △s would have two sides and the ∠ between them in the one, equal respectively to two sides and the ∠ between them in the other.

Consequently (Prop. IV.) these △s would also be equal in every other respect.

Among these respects would be that the ∠ L B A would be equal to the ∠ F E D.

Now the ∠ F E D is equal to the ∠ C B A. (This is one of the points supposed at first respecting the △s A B C and D E F.)

Therefore also the ∠ L B A would be equal to the ∠ C B A:—That is to say, a part would be equal to the whole of which it is a part—which is an impossibility.

It follows, therefore, that the supposition which would lead of necessity to this impossibility (namely, that the side A C is greater than the side D F) must itself be impossible. That is to say, the side A C is not greater than the side D F.

In a similar manner, it may be shown that the side A C is not less than the side D F.[†]

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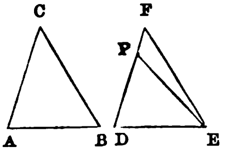
[*] This we know to start with.

[†] The proof of this at length is as follows:—



Suppose it possible that D F should be longer than A C, we should then be able to cut off a part, D P, equal to A C. If this were done, and the points P and E were joined by the right line P E, we should have a △ P D E, in which the side P D would be equal to the side C A, the side D E to the side A B, and the ∠ P D E to the ∠ C A B.

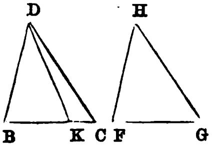
But the line A C must be either greater than D F, less than D F, or equal to D F.

It follows, therefore, since A C can be neither greater nor less than D F, that it must be equal to D F.

Hence the two △s, C A B and F D E, have two sides of the one (namely C A and A B) equal respectively to two sides of the other (namely F D and D E), and the included ∠ C A B in the first equal to the included ∠ F D E in the second.

It follows, therefore (Prop. IV.), that these △s are also equal in every other respect.

Case 2. Next suppose that the △s B C D and F G H have the ∠ D B C in the one equal to the ∠ H F G in the other, the ∠ D C B in the one equal to the ∠ H G F in the other, and the sides B D and F H (opposite the equal ∠s B C D and F G H) equal to one another.



In the proof of this case also the first step consists in

Consequently (according to the fourth proposition) these △s would be equal in every respect.

Among these respects would be that the ∠ P E D would be equal to the ∠ C B A.

But it was supposed, to start with, that the ∠ C B A is equal to the ∠ F E D.

Therefore, the ∠ P E D would also be equal to the ∠ F E D.

But this is impossible, for the ∠ P E D is only a part of the ∠ F E D.

It follows, therefore, that the supposition which led to this absurdity must itself be impossible.

That is, it is impossible that the side D F should be longer than the side A C, or in other words that the side A C should be less than the side D F.

showing that the △s have one other pair of sides equal, namely, the pair consisting of that side in each △ which, with one of the sides supposed to be equal, includes, in each △, one of the equal ∠s (or the pair of sides to be proved equal may be described as being opposite that pair of ∠s which are *not* known or supposed to be equal, to start with). In the △s before us these sides must be B C and F G, which with the equal sides B D and F H, include the equal ∠s D B C and H F G.*

If B C and F G were not equal, one or other would be the greater of the two.

Suppose the side B C were longer than the side F G, we should then be able to cut off a part, B K, equal to F G.

If this were done, and the points D and K were joined by the right line D K, we should get a △, D B K, in which the side D B would be equal to the side H F, the side B K to the side F G, and the ∠ D B K to the ∠ H F G. That is to say, two sides and the included ∠ in the one △ would be equal respectively to two sides and the included ∠ in the other △.

It would follow (according to the fourth proposition) that these △s would be equal also in every other respect.

Among these respects would be, that the ∠ D K B would be equal to the ∠ H G F.

Now it was supposed, to start with, that the ∠ D C B is equal to the ∠ H G F.

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* It would be of no use to show that the sides D C and H G are equal, even if it were possible to do so, for we should still be unable to apply the fourth proposition to prove that the △s are equal in every other respect. For though we should have two sides in the one, viz., B D and D C, equal to two sides in the other, viz. F H and H G, we are not entitled to affirm (before it is proved), that the ∠ B D C is equal to the ∠ F H G.

It would follow, therefore, that the ∠ D K B would be equal to the ∠ D C B.

But this is impossible, for the ∠ D K B is the exterior ∠ of the △ D C K, formed by the production of the side C K, and is therefore greater than the ∠ D C B.

Consequently the supposition which led to this impossibility must be itself impossible. That is to say, it is imposible that the side B C should be greater than the side F G.

In the same way it may be proved that the side B C is not less than the side F G.*

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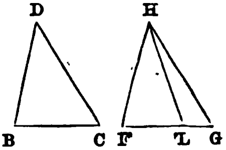
* The proof, at length, is as follows:—



If F G were greater than B C, we should be able to cut off a part, F L, equal to B C.

If this were done, and the points H and L were joined by the straight line, H L, we should have a △ H F L, in which the side H F would be equal to the side D B in the △ D B C, the side F L to the side B C, and the included ∠ H F L to the included ∠ D B C. Consequently (according to the fourth proposition), the △s H F L and D B C would also be equal in every other respect.

Among these respects would be that the ∠ H L F would be equal to the ∠ D C B.

Now the ∠ D C B we know, to start with, to be equal to the ∠ H G F.

Consequently, the ∠ H L F would be equal to the ∠ H G F.

But this is impossible, for H L F is the exterior ∠ of the △ H G L, formed by the production of the side G L, and is therefore greater than the interior and opposite ∠, H G F.

It follows, therefore, that the supposition which led to this impossibility must itself be impossible, that is to say, it is impossible that the side F G should be greater than the side B C.

But the side B C must be either greater than, less than, or equal to, the side F G.

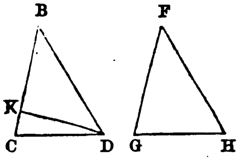
Consequently, as it is neither greater nor less than F G, it is equal to F G.

Hence, in the △s D B C and H F G, two sides of the one, namely D B and B C, are equal to two sides of the other, namely H F and F G, each to each, and the included ∠ D B C of the one is equal to the included ∠ H F G of the other.

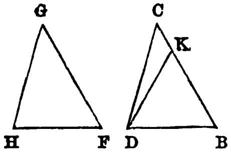
Consequently, according to the fourth proposition, these △s are also equal in every other respect.

This case of the proposition may next be proved, supposing that the angles D B C and D C B are equal to the ∠s H F G and H G F, each to each, and the side D C to the side H G (in the last figure).

By altering the position of the letters to that indicated in the following diagram, the demonstration of the proposition will remain word for word the same.



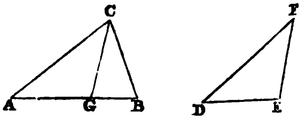
As another variation let the proposition be gone through with the changes which will be occasioned by placing the letters as in the following diagram :—



The 4th, 8th, and 26th propositions contain all the conditions essential for the equality of two △s in all respects, with one exception, which will be mentioned presently. It will be seen that *three* parts of the one must be equal to *three* parts of the other respectively, one at least of these three parts being a side of the △. The parts that they must have respectively equal, are—1. Two sides and the ∠ between them. (IV.) 2. Three sides. (VIII.) 3. Two ∠s and one side similarly placed with respect to the equal ∠s. (XXVI.)

If two △s have two sides of the one equal to two sides of the other, each to each, and *one* ∠ in the one equal to *one* ∠ in the other, opposite corresponding equal sides, the △s are not necessarily equal in all respects. But the ∠s opposite the other pair of equal sides will be either equal or supplementary.*

In the △s A B C and D E F, let the side A C be equal to the side D F, the side C B to the side F E, and the ∠ C A B to the ∠ F D E. If the ∠s A C B and D F E are also equal, the △s are equal in every respect. If not, let one of them, A C B, be larger than the other, and from C draw the line C G, making with the longer of the two sides A C, C B an ∠ A C G, equal to the ∠ D F E, and meeting A B in G. The △s A C G and D F E will be equal to each other, in every respect (according to the 26th proposition). Consequently, the ∠ A G C is equal to the ∠ D E F. But C G is equal to F E, and is therefore equal to C B. Consequently the ∠s C G B and C B G are equal to each other.



But C G A and C G B together make up two right ∠s; therefore C B A and F E D together make up two right ∠s.

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* Two ∠s are said to be *supplementary* when they are together equal to two right ∠s. They are said to be *complementary*, when they are together equal to one right ∠.

If anything enables us to affirm that the ∠s C B A and F E D cannot be supplementary to each other, it will follow that they are equal, and therefore that the △s are equal in all respects. If we know that they are both obtuse, they cannot be supplementary, because the supplement of an obtuse ∠ must be acute.

If we know that the ∠s which are given equal are either right or obtuse, the ∠s opposite the other pair of equal sides must be acute (Prop. XXXII.), and therefore cannot be supplementary to each other.

If the ∠s which must be either equal or supplementary are opposite sides which are *shorter* than the other pair of equal sides, these ∠s cannot be supplementary; for if they were, one of them would be obtuse, and an obtuse ∠ in a △ must be the largest of the three ∠s (Prop. XXXII.), and therefore must have the longest side opposite to it. (Prop. XIX.)

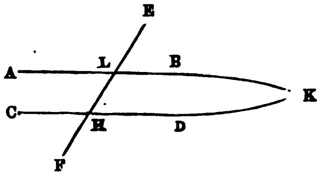
If we know that one of the two ∠s opposite the second pair of equal sides is a right ∠, the other must also be a right ∠, because a right ∠ is the supplement of a right ∠; therefore it comes to the same thing whether they are considered as equal or as supplementary.

PROPOSITION XXVII.

If a straight line meets two other straight lines, and the alternate angles so formed are equal, those straight lines are parallel,—that is, will never meet one another, however far they may be produced.

To prove this proposition, we must know, that if one side of a △ be produced, the exterior ∠ is greater than either of the two interior and opposite ∠s. (Prop. XVI.)

The proof of this proposition is of the indirect kind.



A B and C D are two straight lines intersected by the straight line E F, and the alternate ∠s A L H and L H D are equal to one another.* We have to prove that the lines A B and C D when produced will not meet.

Proof. Suppose it were possible that they should meet at a point K. The two lines L K and H K, with the line L H, would then form a △ L K H.

The ∠ A L H would be the exterior ∠ of this △, formed by the production of the line K L, and would (Prop. XVI.) be greater than the interior and opposite ∠ L H D.

But this is impossible, for the ∠s A L H and L H D are equal.

Therefore the supposition that would lead to this impossibility must be itself impossible. That is, it is impossible that the straight lines A B and C D should meet, when produced, in the point K.

In a similar way it might be shown that they could not meet, when produced, on the other side of the line E F.

In other words (see Def. XXVI.) the straight lines A B and C D are parallel.

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* The ∠s B L H and L H C are also alternate ∠s. If we know either pair of ∠s to be equal, it will follow that the other pair are also equal; for the ∠s A L H and B L H are together equal to two right ∠s, as are also the ∠s L H D and L H C, and consequently the ∠s B L H and A L H are together equal to L H D and L H C. If then either pair of alternate ∠s be equal, and be taken away from these equal sums, the remainders will be equal.

PROPOSITION XXVIII.

If a straight line intersects two other straight lines, and the exterior angle so formed is equal to the interior and opposite angle on the same side of the intersecting line, or if the two interior angles on either side of the intersecting line are together equal to two right angles, then those two straight lines are parallel.

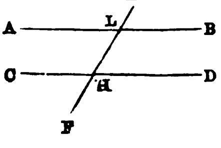
The above enunciation really includes two separate propositions, which will be more clearly understood if taken separately.

1. If a straight line intersects two other straight lines, and the exterior angle so formed is equal to the interior and opposite angle on the same side of the intersecting line, then those two straight lines are parallel (i.e., will not meet, however far they may be produced).

To prove this we must know,—

1. That magnitudes which are equal to the same are equal to one another. (Ax. I.)
2. That if two straight lines intersect one another, the vertically opposite $\angle$s are equal. (Prop. XV.)
3. That if one straight line meet two other straight lines, and make the alternate $\angle$s equal, those straight lines are parallel. (Prop. XXVII.)

The straight lines A B and C D are intersected by the line E F, and the exterior $\angle$ E L B is equal to the interior and opposite $\angle$ L H D on the same side of the intersecting line. We have to prove that the straight lines A B and C D are parallel.



Proof. The ∠s E L B and A L H, being the vertically opposite ∠s formed by the intersection of the lines A B and E F are equal. (Prop. XV.)

But the ∠ E L B is equal to the ∠ L H D.

Therefore the ∠ A L H is also equal to the ∠ L H D.

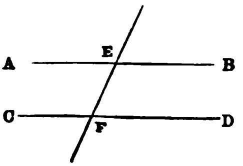
But A L H and L H D are alternate ∠s. Consequently, as the line E F intersects the lines A B and C D, and the alternate ∠s are equal, the lines A B and C D are parallel. (Prop. XXVII.)

Similarly, if, instead of the ∠s E L B and L H D, it were known that the ∠s E L A and L H C were equal, the proposition would be proved by showing that the ∠ E L A is equal to the vertically opposite ∠ B L H, and that consequently the alternate ∠s B L H and L H C would be equal.

2. If a right line meets two other right lines, and the two interior angles are together equal to two right angles, those right lines are parallel.

To prove this proposition we must know:—

1. That if one straight line meets another straight line, the adjacent ∠s so formed are together equal to two right ∠s. (Prop. XIII.)
2. That things that are equal to the same are equal to one another. (Ax. I.)
3. That if the same quantity be taken from two equal quantities the remainders are equal. (Ax. III.)
4. That if one straight line meets two other straight lines, and the alternate ∠s so formed are equal, those straight lines are parallel. (Prop. XXVII.)



A B and C D are two straight lines met by the straight line E F, and the ∠s BEF and EFD are together equal to two right ∠s. It has to be proved that the right lines A B and C D are parallel.

Proof. Because the straight line F E meets the straight line A B, the adjacent ∠s A E F and B E F are together equal to two right ∠s. (Prop. XIII.)

But the ∠s B E F and E F D are also together equal to two right ∠s.

Therefore the sum of the ∠s A E F and B E F is equal to the sum of the ∠s B E F and E F D.

If from each of these equal sums we take away the ∠ B E F, the remainders are equal. That is to say, the ∠ A E F is equal to the ∠ E F D.

But A E F and E F D are alternate ∠s.

Therefore, as these alternate ∠s are equal, the lines A B and C D will never meet when produced; *or in other words are parallel.* (Prop. XXVII.)

Thus far there is no difficulty whatever with respect to parallel lines. The proof of the next proposition, however, requires that we should assume the truth of the 11th axiom. Now this axiom, though no one ever doubted its truth, is by no means so obviously true as the other axioms. In fact there are several propositions of which a detailed proof is given in geometry, the truth of which is more readily perceived than that of the 11th axiom. Numerous attempts have, in consequence, been made to supersede the necessity for the use of this axiom; but entirely without success, as some axiom or other is always assumed which is quite as difficult as the 11th. The difficulty, of course, arises from the fact that the definition of parallel lines is of a *negative* character, and therefore, without some *positive* axiom, would not enable us to found any inferences upon it.

PROPOSITION XXIX.

If two straight lines are parallel and a third straight line intersects them, then the alternate angles so formed will be equal, the exterior angle will be equal to the interior and opposite angle on the same side of the intersecting line, and the two interior angles on the same side of the intersecting line will together be equal to two right angles.

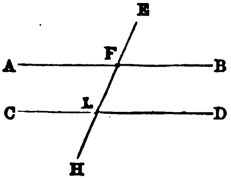
For the proof of this proposition we must know,—

1. That if two straight lines meet a third in such a way that the two interior ∠s on the same side of it are together less than two right ∠s, those straight lines will meet if produced. (Ax. XI.)
2. That magnitudes that are equal to the same are equal to one another. (Ax. I.)
3. That if equals or the same be taken from equals, the remainders are equal. (Ax. III.)
4. That if one straight line meets another, the adjacent ∠s so formed are together equal to two right ∠s. (Prop. XIII.)
5. That if two straight lines intersect one another, the vertically opposite ∠s are equal. (Prop. XV.)

The 29th proposition in reality consists of three distinct propositions, classed together under one heading or enunciation, and corresponding respectively to the 27th and the two parts of the 28th proposition, of which they are the converse.

As all three cases of the 29th proposition depend upon the application of the 11th axiom, it will be most convenient to take the third case first, as in it the axiom in question is applied most directly.

1. If two straight lines are parallel, and a third meets them, the two interior angles on the same side of the intersecting line are together equal to two right angles.



The two lines A B and C D are parallel (*i.e.*, however far they may be produced, they will never meet). The line E H intersects them. We have to prove that the ∠s B F L and F L D are together equal to two right ∠s.

The proof of this is of the indirect kind. If the ∠s B F L and F L D are not together equal to two right ∠s, they must be either less than two right ∠s, or greater than two right ∠s.

If they were less than two right ∠s, then the lines F B and L D would meet if produced (according to the 11th axiom).

But this is impossible, for we know, to start with, that the lines A B and C D are parallel (*i. e.*, can never meet).

Consequently the supposition that would lead to this impossibility must be itself impossible.

That is to say, it is impossible that the ∠s B F L and F L D should be together less than two right ∠s.

Again—suppose the ∠s B F L and F L D were together greater than two right ∠s. Since the ∠s B F L and L F A are together equal to two right ∠s (according to Prop. XIII.), and the ∠s F L D and F L C are also equal to two right ∠s, the four ∠s B F L, L F A, F L D, and F L C are together equal to four right ∠s. Therefore, it would follow that on taking away from the sum of these four ∠s the two ∠s B F L and F L D, if the latter were together greater than two right ∠s, the remaining ∠s A F L and F L C would be together less than two right ∠s.

But if the two straight lines A F and C L meeting a third, F L, made the interior ∠s A F L and F L C together less than two right ∠s, the two straight lines F A and L C would meet if produced. (Ax. XI.)

But this is impossible, because we know, to start with, that F A and L C are parallel.

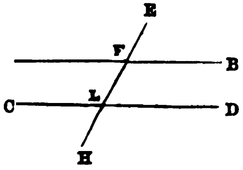
Consequently, it is impossible that the two ∠s A F L and F L C can be together less than two right ∠s, and therefore it is impossible that the two ∠s B F L and F L D should be together greater than two right ∠s.

Hence, as the ∠s B F L and F L D can be neither less than two right ∠s nor greater than two right ∠s, they must together be equal to two right ∠s.

Moreover, since B F L and F L D are together equal to two right ∠s, if they be taken away from the four ∠s B F L, L F A, F L D, and F L C, the remaining ∠s L F A and F L C will be together equal to two right ∠s.

2. If two straight lines are parallel and a third straight line intersects them, the alternate angles so formed will be equal.

Let A B and C D be parallel, and let E H intersect them in the points F and L. Then the ∠ A F L will be equal to the ∠ F L D, and the ∠ B F L to the ∠ F L C.



It has already been proved that the two interior ∠s B F L and F L D are together equal to two right ∠s.

Also, since the straight line L F meets the straight line A B, the adjacent ∠s A F L and B F L are together equal to two right ∠s. (Prop. XIII.)

Therefore the sum of the ∠s B F L and F L D is equal to the sum of the ∠s A F L and L F B. (Ax. I.)

Take away the common ∠, B F L, and the remainders, the ∠s A F L and F L D will be equal to one another.

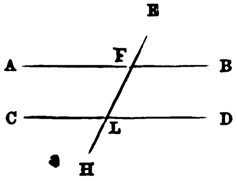
Also, it was proved in the last case that the two interior ∠s A F L and F L C are together equal to two right ∠s.

But the ∠s A F L and B F L are also together equal to two right ∠s. Therefore the ∠s A F L and F L C are together equal to the sum of the two ∠s A F L and L F B.

Take away the common ∠, A F L, and the remainders, namely, the ∠s B F L and F L C, will be equal.

3. If two straight lines are parallel and a third straight line intersects them, each of the exterior angles is equal to the interior and opposite angle on the same side of the intersecting line.

Let A B and C D be parallel, and let E H intersect them in the points F and L. We have to prove that the ∠ E F B is equal to the ∠ F L D, and the ∠ E F A to the ∠ F L C.



Proof. Since the straight lines A B and E H intersect one another at the point F, the vertically opposite ∠s E F B and A F L are equal. (Prop. XV.)

But it was proved in the last case that the ∠ A F L is equal to the ∠ F L D.

Therefore the ∠ E F B is also equal to the ∠ F L D.

Again, the vertically opposite ∠s E F A and B F L are equal. (Prop. XV.)

But it was proved that the ∠ B F L is equal to the ∠ F L C.

Therefore the ∠ E F A is also equal to the ∠ F L C.

In a similar way it might be shown that the ∠ H L D is equal to the ∠ L F B, and the ∠ H L C to the ∠ L F A.

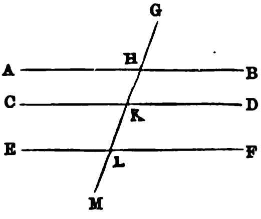
PROPOSITION XXX.

Straight lines which are parallel to the same straight line are parallel to each other.

To prove this proposition we must know,—

1. That magnitudes which are equal to the same are equal to one another. (Ax. I.)
2. That if two straight lines are parallel, and are cut by a third straight line, the alternate ∠s so formed are equal. (Prop. XXIX.)
3. That if two straight lines are parallel, and are cut by a third, the exterior ∠ is equal to the interior and opposite ∠ on the same side of the intersecting line. (Prop. XXIX.)
4. That if two straight lines are crossed by a third, and the exterior ∠ so formed is equal to the interior and opposite ∠ on the same side of the intersecting line, those straight lines are parallel. (Prop. XXVIII.)

Case 1. Suppose the two straight lines A B and C D are each parallel to E F. We have to prove that A B and C D are parallel to each other.



Proof. Draw the straight line G M, intersecting the lines A B, C D, and E F, in the points H, K, and L. Because the lines A B and E F are parallel and the line G M cuts them, the exterior ∠ G H A is equal to the interior and opposite ∠ H L E.

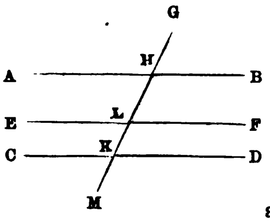
Again, because C D and E F are parallel, the exterior ∠ H K C is equal to the interior and opposite ∠ K L E.

Therefore, since the ∠s G H A and H K C are both equal to the ∠ K L E, they are equal to each other.

Now the ∠s G H A and H K C are the exterior and interior opposite ∠s, formed by the intersection of the two lines A B and C D by the line G M.

But if two straight lines are crossed by a third, and the exterior ∠ so formed is equal to the interior and opposite ∠ on the same side of the intersecting line, those straight lines are parallel. (Prop. XXVIII.)

Therefore the lines A B and C D are parallel.



Case 2. Suppose the lines A B and C D are both parallel to the line E F (which lies between them).

Draw the line G M intersecting the three lines in the points H, K, and L.

Because the lines A B and E F are parallel, and the line G M intersects them, the exterior ∠ G H A is equal to the interior and opposite ∠ H L E.

Again, because E F and C D are parallel, and the line G M intersects them, the exterior ∠ H L E is equal to the interior and opposite ∠ L K C.

Therefore, since the ∠s G H A and L K C are both equal to the ∠ H L E, they are equal to each other.

We have, therefore, two straight lines, A B and C D, cut by a third, so that the exterior ∠ thus formed (namely G H A) is equal to the interior and opposite ∠ on the same side of the intersecting line (namely, the ∠ H K C).

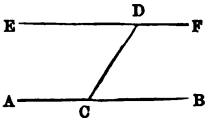
It follows, therefore, according to the 28th proposition that these two right lines A B and C D are parallel to each other.

PROPOSITION XXXI.

To draw a straight line through a given point parallel to a given straight line.

> For the construction requisite in this proposition we must be able,—
>
> 1. To join two given points by a straight line. (Post. I.)
> 2. From a given point in a given straight line to draw a straight line making with the given line an ∠ equal to a given ∠. (Prop. XXIII.)
>
> To prove that the construction effects what is required we must know,—
>
> That if two straight lines are met by a third, and the alternate ∠s so formed are equal to one another, those two straight lines are parallel. (Prop. XXVII.)

Let A B be the given straight line, and D the given point. We have to draw through the point D a line parallel to the given line A B.



In the line A B take any point, as C.

Join D and C by the straight line D C.

From the point D draw the straight line D E, making with the line D C an ∠, C D E, equal to the ∠ D C B. (Prop. XXIII.)

Produce the line E D through the point D to any point F. E F is the line required.

For since the two straight lines E F and A B are met by a third straight line, and the alternate ∠s so formed. namely, C D E and D C B, are equal, the lines E F and A B are parallel. (Prop. XXVII.)

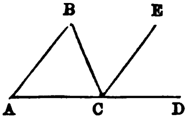
PROPOSITION XXXII.

If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior and opposite angles; and the three interior angles of a triangle are together equal to two right angles.

For the construction employed in this proposition we must be able to draw through a given point a right line parallel to a given right line. (Prop. XXXI.)

For proving the proposition by the aid of the construction we must know,—

1. That if the same, or equal quantities be added to equals, the sums will be equal. (Ax. II.)
2. That if one straight line meet another, the adjacent ∠s so formed are together equal to two right ∠s. (Prop. XIII.)
3. That if two straight lines are parallel and a third straight line meets them, the alternate ∠s so formed are equal, and the exterior ∠ is equal to the interior and opposite ∠ on the same side of the intersecting line. (Prop. XXIX.)



Let A B C be a △, of which one side, A C, has been produced.

It has to be shown that the exterior ∠ B C D is equal to the sum of the two interior and opposite ∠s, B A C and A B C, and that the three interior ∠s of the △ A B C are together equal to two right ∠s.

Through the point C draw the straight line C E parallel to the side A B.

The exterior ∠ B C D is thus divided into two parts; and the proof of the proposition consists in showing that one of these parts is equal to one of the interior and opposite ∠s; and the other part to the other.

Since the lines A B and C E are parallel and are met by a third line, B C, it follows (Prop. XXIX.), that the alternate ∠s so formed, namely A B C and B C E, are equal.

Again, since A B and C E are parallel, and are intersected by a third line, A D, it follows that the exterior ∠ E C D is equal to the interior and opposite ∠ on the same side of the intersecting line, namely the ∠ B A C.

But if equals be added to equals, the sums are equal. (Ax. II.)

Therefore, if the equal ∠s A B C and B C E be added respectively to the ∠s B A C and E C D, the sums will be equal: that is, the sum of the ∠s B C E and E C D will be equal to the sum of the ∠s A B C and B A C.

B C E and E C D together make up the exterior ∠ B C D.

Therefore the exterior ∠ B C D is equal to the sum of the two interior and opposite ∠s, A B C and B A C.

Now, if to each of these equals be added the ∠ B C A, the sums will be equal. (Ax. II.)

That is to say, the sum of the exterior ∠ B C D, and the ∠ B C A, will be equal to the sum of the ∠s A B C, B A C, and B C A.

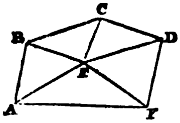
But since the straight line B C meets the straight line A D, the sum of the adjacent ∠s B C D and B C A is equal to two right ∠s.

Therefore, also, the sum of the three interior ∠s of the △ A B C is equal to two right ∠s.

In the same way it may be proved that the sum of the three interior ∠s of any other △ is equal to two right ∠s.

This important proposition involves an immense variety of consequences. Some of the most obvious are stated in the following corollaries :—

1. If one ∠ of a △ is a right ∠, the other two ∠s are together equal to a right ∠.
2. If one ∠ of a △ is equal to the sum of the other two ∠s, that ∠ is a right ∠.
3. An obtuse ∠ is greater, and an acute ∠ is less than the sum of the other two ∠s.
4. If one ∠ of a △ is greater than the sum of the other two, it is obtuse; if it is less than the sum of the other two, it is acute.
5. If two isosceles △s have their vertical ∠s equal, the ∠s at their bases are also equal.
6. If the vertical ∠ of an isosceles △ is a right ∠, each of the ∠s at the base is half a right ∠.
7. The sum of all the internal ∠s of any rectilinear figure, together with four right ∠s, is equal to twice as many right ∠s as the figure has sides. Let A B C D E be a rectilinear figure. Take a point F somewhere inside the figure, and join the points F and A, F and B, F and C, F and D, F and E.

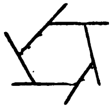


The whole figure is thus divided into as many △s as the figure has sides.

The sum of the three ∠s of each of these △s is two right ∠s; consequently, the sum of all the ∠s of all the △s is twice as many right ∠s as the figure has sides.

The ∠s at the point F together make up four right ∠s; the other ∠s of the △s make up the ∠s of the polygon A B C D E. Therefore the ∠s of the polygon together with four right ∠s are equal to twice as many right ∠s as the figure has sides.

8. If the sides of any convex rectilinear figure be produced, the external ∠s are together equal to four right ∠s. For each external ∠ with the internal ∠ adjacent to it, makes up the two right ∠s. Consequently, all the internal ∠s and all the external ∠s taken together make up twice as many right ∠s as the figure has sides. But all the internal ∠s, together with four right ∠s, make up twice as many right ∠s as the figure has sides. It follows, therefore, that all the external ∠s taken together, make up four right ∠s.



9. Each ∠ of an equilateral △ is the third part of two right ∠s, or two-thirds of one right ∠. Therefore, if we bisect an ∠ of an equilateral △, we get the third part of a right ∠. (See note on Prop. IX.)

PROPOSITION XXXIII.

If there are two straight lines which are equal and parallel to each other, and their corresponding[*] extremities be joined by straight lines, these straight lines will also be equal and parallel.

For the construction in this proposition we must be able to join two given points by a straight line.

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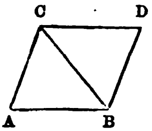
[*] By *corresponding extremities* are meant those which point in the same direction.

To prove the proposition by the help of the construction we must know,—

1. That if two △s have two sides of the one equal respectively to two sides of the other, and also the ∠ included between the said sides in the first △ equal to the ∠ included between the said sides in the other △, those △s will also be equal to each other in every other respect. (Prop. IV.)
2. That if two straight lines are parallel and a third line meets them, the alternate ∠s so formed are equal. (Prop. XXIX.)
3. That if the alternate ∠s, formed when two straight lines are met by a third, are equal, those two straight lines will be parallel. (Prop. XXVII.)

Suppose C D and A B to be two equal and parallel straight lines, and let the corresponding extremities C and A and D and B be joined by the straight lines C A and D B.



It has to be proved that the lines C A and D B are equal and parallel.

Join by a straight line either pair of extremities of the equal and parallel straight lines, which do *not* correspond, as, for example, C and B.

We have thus formed two △s, C D B and B A C.

In these △s the side C D is equal to the side B A. (This we know to start with.)

The side C B is common to the two △s.

Also the ∠s D C B and C B A are equal, because they are the alternate ∠s formed by the parallel straight lines C D and A B when met by a third line, C B.

Thus the two △s C D B and B A C have two sides of the one (namely D C and C B) equal respectively to two sides of the other (namely A B and B C), and the included ∠ D C B of the one equal to the included ∠ A B C of the other.

Consequently (according to the fourth proposition), these triangles are also equal in every other respect.

Among these respects are :—

1. That the side B D is equal to the side A C. (This is one part of the proposition to be proved.)

2. That the ∠ D B C is equal to the ∠ A C B.

Now D B C and A C B are the alternate ∠s formed by the two straight lines A C and B D when met by a third line, C B.

Hence, since these alternate ∠s are equal, the lines A C and B D are parallel. (Prop. XXVII.)

It has thus been shown that the lines A C and B D are both equal and parallel.

PROPOSITION XXXIV.

The opposite sides and angles of a parallelogram are equal to each other, and a parallelogram is bisected (that is, divided into two equal parts) by each of its diagonals.

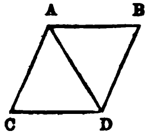
To prove this proposition we must know,—

1. That if equals be added to equals the sums are equal. (Ax. II.)
2. That if two △s have two sides and the ∠ between them in the one, equal respectively to two sides and the ∠ between them in the other, those △s are also equal in every other respect. (Prop. IV.)
3. That if two △s have two ∠s of the one equal respectively to two ∠s of the other, and a side of the first △ equal to a side of the

second, similarly placed with respect to the equal ∠s, those △s are also equal in every other respect. (Prop. XXVI.)

4. That if two lines are parallel and a third line meets them, the alternate ∠s so formed are equal. (Prop. XXIX.)

Let A B D C be a parallelogram, that is, suppose it to be a four-sided figure, whose opposite sides are parallel, and let A D be one of its diagonals.



We have to prove that the opposite sides are equal:—namely, the side A B to the side C D, and the side A C to the side B D: and also that the opposite ∠s are equal, namely, the ∠ A B D to the ∠ A C D, and the ∠ C A B to the ∠ B D C and that the parallelogram is divided into two equal parts by its diagonal A D.

Proof. Because A B and C D are parallel, and the line A D meets them, it follows that the alternate ∠s so formed, namely, B A D and A D C, are equal. (Prop. XXIX.)

Because A C and B D are parallel and the line A D meets them, it follows (Prop. XXIX.) that the alternate ∠s so formed, namely, C A D and A D B, are equal.

Hence, the two △s A B D and C D A have the two ∠s B A D and B D A of the one equal respectively to the two ∠s A D C and C A D of the other, and the side A D common to the two △s and similarly placed with respect to the equal ∠s.

It follows, therefore (Prop. XXVI.), that these △s are also equal in every other respect. That is to say, the ∠ A B D is equal to the ∠ A C D; the side A B to the side C D, the side B D to the side A C, and the area of the one △ to the area of the other.

Moreover, since it was shown that the two parts B A D and C A D of the ∠ B A C are equal respectively to the two parts, A D C and B D A, of the ∠ B D C, if the one pair of equal ∠s be added to the other the sums will be equal; that is to say, the ∠ B A C is equal to the ∠ B D C.

It has thus been shown that in the parallelogram A B D C the opposite sides are equal—namely, A B to C D, and A C to B D; and that the opposite ∠s are equal, namely, the ∠ A B D to the ∠ A C D, and the ∠ B A C to the ∠ B D C; and that the diagonal A D divides the parallelogram into two △s which are equal in area.

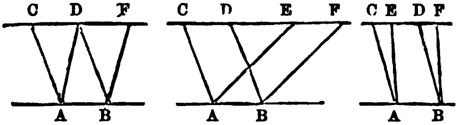
PROPOSITION XXXV.

Parallelograms upon the same base and between the same parallels are equal to one another.

For the proof of this proposition we require to know,—

1. That if two straight lines are parallel and are intersected by a third straight line, the exterior ∠ so formed is equal to the interior and opposite ∠ on the same side of the intersecting line. (Prop. XXIX.)
2. That if two △s have two ∠s of the one equal to two ∠s of the other, each to each, and likewise a side of the first equal to a side of the second, similarly placed with respect to the equal ∠s, those △s will also be equal in every other respect. (Prop. XXVI.)
3. That if equal magnitudes be taken from the same magnitude, the magnitude remaining in the one case will be equal to the magnitude remaining in the other. (Ax. III.)

(The following demonstration is applicable to any one of the three figures given, which indicate the different positions which the parallelograms may occupy with relation to each other, with this exception, that in applying the demonstration to the first figure, wherever the letter E occurs, the letter D must be substituted for it. The second and third figures had better be employed first in going through the proposition. The peculiarity of the first figure is that the points D and E coincide.)



Let A B D C, and A B F E be two parallelograms upon the same base, A B, and between the same parallels, A B and C F. It has to be shown that they are equal in area.

To demonstrate this it is shown: 1st. That the △s C A E and D B F are equal in area. 2nd. That the remainder left, when the △ C A E is taken from the trapezium C A B F, is equal to the remainder left when the △ B D F is taken away from the same figure.

Since the straight lines C A and D B are parallel and the straight line C F intersects them, the exterior ∠ F D B is equal to the interior and opposite ∠ on the same side of the intersecting line, namely, E C A. (Prop. XXIX.)

Again, since the straight lines A E and B F are parallel, and the straight line C F intersects them, the exterior ∠ C E A is equal to the interior and opposite ∠ on the same side of the intersecting line—namely, D F B.

And because C A B D is a parallelogram, its opposite sides A C and B D are equal.

Hence it appears that the two △s C A E and D B F have two ∠s of the one (namely A C E and A E C) equal respectively to two ∠s of the other (namely B D F and B F D), and a side of the one (viz. A C) equal to a side of the other similarly placed with respect to the equal ∠s (namely B D).

These △s are therefore equal in every other respect; and, among these respects, in area.

Now if the △ C A E be taken away from the figure A B F C, we have left the parallelogram A B F E; and if the △ D B F be taken away from the same figure A B F C, we have left the parallelogram A B D C.

But since the △s are equal in area, the remainder in the first case must be equal to the remainder in the second. That is to say, the parallelogram A B F E is equal in area to the parallelogram A B D C.

Let this proposition next be gone through with the figures drawn in an inverted position, and then with the use of different letters.

The altitude of a parallelogram is the length of a perpendicular drawn from the base on which it stands to the opposite side, or the opposite side produced.

It follows from the 28th proposition that perpendiculars to the same right line are parallel—the two interior angles being together equal to two right ∠s.

It follows accordingly from the 33rd proposition that if two perpendiculars to the same right line are of equal length, the line joining their extremities is parallel to the line on which they stand.

It also follows from the 34th proposition that all perpendiculars drawn between two parallel straight lines are of the same length.

Hence it appears that if two parallelograms are between the same parallels, they may also be described as having the same altitude. And *vice versâ*, that if two parallelograms have their bases in the same straight line, and have the same altitude, they may also be described as being between the same parallels.

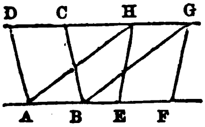
PROPOSITION XXXVI.

Parallelograms upon equal bases and between the same parallels (or having the same altitude) are equal to one another.

For the construction in this proposition we must be able to join two given points by a straight line.

For the demonstration of the proposition we must know,—

1. That the opposite sides of parallelograms are equal. (Prop. XXXIV.)
2. That the lines joining the corresponding extremities of equal and parallel straight lines are themselves equal and parallel. (Prop. XXXIII.)
3. That parallelograms upon the same base and between the same parallels are equal. (Prop. XXXV.)
4. That things that are equal to the same are equal to each other. (Ax. I.)



Let A B C D and E F G H be two parallelograms upon equal bases (A B and E F), and between the same parallels. It has to be shown that they are equal to each other in area. Join the points A and H by the straight line A H, and the points B and G by the straight line B G.

We thus get a quadrilateral figure A B G H, which it is easy to show is a parallelogram. For, since the figure E F G H is a parallelogram, the opposite sides E F and G H are equal,

Now E F is by hypothesis equal to A B.

Therefore A B is also equal to H G.

Moreover, the lines D G and A F are supposed to be parallel to start with.

Consequently, the lines A B and H G are both equal and parallel.

It follows, therefore (according to the 33rd proposition), that the lines A H and B G, which join their corresponding extremities, are also parallel; so that the figure A B G H is a parallelogram.

Now the parallelograms A B C D and A B G H are upon the same base, A B, and between the same parallels, A B and D G. They are therefore equal in area. (Prop. XXXV.)

Again, the parallelograms H G B A, and H G F E are upon the same base, H G, and between the same parallels, H G and A F. They are therefore equal in area. (Prop. XXXV.)

But things that are equal to the same are equal to each other. (Ax. I.)

Hence the parallelograms A B C D and E F G H, being each equal to the parallelogram A B G H, are equal to each other.

PROPOSITION XXXVII.

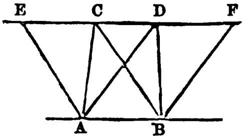
Triangles upon the same base and between the same parallels are equal to each other.

For the construction in this proposition we must be ableto draw a straight line from a given point, so as to be parallel to a given straight line. (Prop. XXXI.)

To prove the proposition by the aid of the construction we must know,—

1. That a parallelogram is divided into two equal parts by either of its diagonals. (Prop. XXXIV.)

2. That parallelograms upon the same base and between the same parallels are equal. (Prop. XXXV.)
3. That the halves of equals are equal. (Ax. VIII.)



Let A B C and A B D be two △s upon the same base, A B, and between the same parallels, A B and E F. It has to be shown that they are equal in area.

From the point A draw a line A E, parallel to the side B C of the △ A B C, and meeting the line E F in the point E.

From the point B draw a line, B F, parallel to the side A D, of the △ A B D, and meeting the line E F in the point F.*

We thus get two parallelograms, A B C E and

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* It will answer the purpose just as well, if from the point A a line be drawn parallel to the line B D, and from the point B a line parallel to the line A C. All that we have to attend to is, that the line drawn from the point A be parallel to a side of one of the △s, and that the line drawn from the point B be a parallel to a side of *the other* △.

It would also do equally well if from the point A *two* lines were drawn, one parallel to a side of one △, and the other parallel to a side of the other. Or, in like manner, two lines might be drawn from B, one parallel to a side of one △, the other parallel to a side of the other. The sides to which the lines are to be drawn parallel are of course those which do *not* terminate in the point from which the said lines are drawn. All these varieties of construction effect the same object, viz., to get two parallelograms upon the same base, of which one is the double of one of the △s and the other of the other.

A B F D, upon the same base, A B, and between the same parallels, A B and E F.

These parallelograms, (according to the 35th proposition,) will be equal.

But the parallelogram A B C E is the double of the △ A B C, being divided into two equal parts, of which the △ A B C is one, by its diagonal A C; and the parallelogram A B F D is the double of the △ A B D, being divided into two equal parts, of which the △ A B D is one, by its diagonal, B D. (Prop. XXXIV.)

Consequently, as the halves of equals are equal, the △ A B C is equal to the △ A B D.*

PROPOSITION XXXVIII.

Triangles upon equal bases and between the same parallels are equal.

For the construction in this proposition we must be able to draw a straight line from a given point so as to be parallel to a given straight line. (Prop. XXXI.)

To prove the proposition by the aid of the construction we must know,—

1. That parallelograms upon equal bases and between the same parallels are equal. (Prop. XXXVI.)
2. That a parallelogram is divided into two equal parts by either of its diagonals. (Prop. XXXV.)
3. That the halves of equals are equal. (Ax. VIII.)

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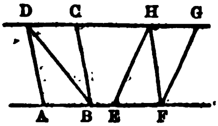
* That is, in area. They have not been shown to be equal in any other respects. Beginners often confound the being equal in area with the being equal in every respect.

Let A B D and E F H be two △s upon equal bases A B and E F, and between the same parallels, A F and D G. It has to be shown that they are equal in area.



From either extremity of A B (as B) draw a line (B C) parallel to that side of the △ A B D which does not terminate in B, and meeting the line D G in the point C.

From either extremity of E F (as F) draw a line (F G) parallel to that side of the △ E F H which does not terminate in F.

We thus get two parallelograms, A B C D and E F G H, which, being upon equal bases and between the same parallels, are equal. (Prop. XXXVI.)

Now the parallelogram A B C D is the double of the △ A B D, being divided into two equal parts, of which the △ A B D is one, by its diagonal B D.

Also the parallelogram E F G H is the double of the △ E F H, being divided into two equal parts, of which the △ E F H is one, by its diagonal F H.

Consequently, since the halves of equals are equal, the △ A B D is equal (in area) to the △ E F H.

PROPOSITION XXXIX.

If two triangles which are equal in area be upon the same base and upon the same side of it, the right line which joins their vertices will be parallel to the base upon which they stand.*

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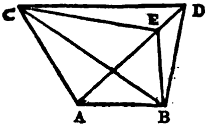
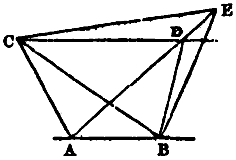
* This proposition is generally stated thus:—"Equal △s upon the same base, and upon the same side of it, are between the same parallels." The meaning of this is just what is stated above, but is less obvious to the beginner.

For the construction in this proposition we must be able to join two given points by a straight line.

For proving the proposition by the aid of the construction, we must know,—

That △s upon the same base and between the same parallels are equal.





The proof of this proposition is of the indirect kind.

Let A B C and A B D be two △s of equal area, upon the same base, A B, and upon the same side of it. It has to be proved that the line C D, which joins their vertices, is parallel to the base, A B.

Proof. If the line C D be not parallel to the base A B, some other line, drawn through C, must be parallel to the base.

Suppose the line C E could be parallel to the base, and suppose C E to meet the side A D, or the side A D produced, in the point E.

Join the points E and B by the straight line E B. We should then have two △s, A C B and A E B, upon the same base, A B, and between the same parallels, A B and C E.

It would follow, therefore, that these △s would be equal to one another. (Prop. XXXV.)

But we know, to start with, that the △ A B C is equal to the △ A B D; therefore the △ A B E would be equal to the △ A B D.

Now this is an impossibility, for the one △ is only a part of the other; consequently, the supposition that leads to this impossibility must be itself impossible.

That is to say, it is impossible that the line C E should be parallel to A B.

In a similar manner it may be shown that no line drawn from the point C can be parallel to the base A B, if it does not coincide with the line C D.

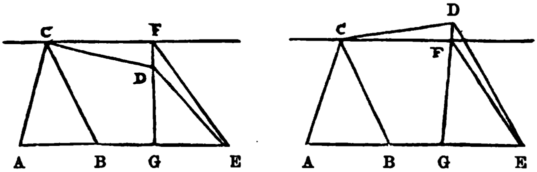
Therefore, as there must be a line, drawn in some direction from C, that is parallel to A B, the line C D must be that line.

PROPOSITION XL.

If two equal triangles stand upon equal bases, which are in the same straight line, the straight line joining the vertices of the triangles will be parallel to the bases.*

For the proof of this proposition we must be able to join two given points by a straight line; and must know that △s upon equal bases and between the same parallels are equal. (Prop. XXXVIII.)

Let A B C and G E F be two △s of equal area, and standing upon equal bases A B and G E, which are in the same straight line. It has to be shown that the line C F, which joins their vertices, is parallel to the line of the bases A B and G E.



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* This proposition is usually stated thus:—Equal △s upon equal bases in the same straight line are between the same parallels.

Proof. If the line C F be not parallel to the bases A B and G E, some other line drawn through C must be parallel to A E.

Suppose that the line C D could be parallel to A E, and suppose C D to meet the side G F, or the side G F produced in D.

Join the points D and E by the right line D E.

We should then have two △s, A B C and G E D, upon equal bases and between the same parallels.

It would follow that these △s must be equal in area. (Prop. XXXVIII.)

Now since the △ A B C is equal in area to the △ G E F, it would follow that the △ G E D would also be equal to the △ G E F.

But this is impossible, because the one △ is only a part of the other.

The supposition which leads to this impossibility must be itself impossible. That is to say, it is impossible that the line C D, which does not coincide in direction with C F, can be parallel to A E.

In the same way it may be proved that no other line which does not coincide with C F can be parallel to A E.

Therefore, as there must be a line drawn in some direction from C that is parallel to A E, the line C F, which joins the vertices of the △s, must be that line.

PROPOSITION XLI.

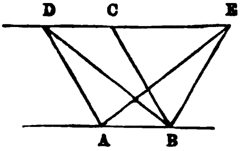
If a parallelogram and a triangle be upon the same base and between the same parallels (or have the same altitude), the parallelogram is the double of the triangle.

For the construction in this proposition we must be able to draw a straight line from one given point to another. (Post. I.)

For the proof of the proposition we must know,—

1. That if two △s be upon the same base and between the same parallels, they are equal. (Prop. XXXVII.)
2. That a parallelogram is bisected by either of its diagonals. (Prop. XXXIV.)
3. That if a magnitude be the double of one of two equal magnitudes, it is also the double of the other (consequence of the 7th axiom).

Let A B C D and A B E be a parallelogram and a △ upon the same base, and between the same parallels. It has to be shown that the parallelogram is the double of the △.



Draw one of the diagonals of the parallelogram, as, *e.g.*, D B.

The △s A B D and A B E are upon the same base and between the same parallels.

It follows (from the 37th proposition) that they are equal to each other (in area).

But the parallelogram A B C D is the double of the △ A B D.

Therefore, it is also the double of the △ A B E, which is equal to the △ A B D.

PROPOSITION XLII.

To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

For the construction necessary in this proposition we must be able,—

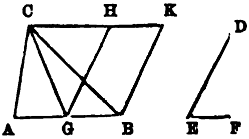
1. To join two given points by a straight line. (Post. I.)

2. To bisect a given straight line.
3. Through a given point to draw a straight line parallel to a given straight line. (Prop. XXXI.)
4. From a given point in a given straight line to draw a straight line making with the given straight line an ∠ equal to a given rectilineal ∠. (Prop. XXIII.)

To prove the proposition by the aid of the construction we must know,—

1. That △s upon equal bases and between the same parallels (or having the same altitude), are equal. (Prop. XL.)
2. That if a parallelogram and a △ be upon the same base, and between the same parallels, the parallelogram is the double of the △. (Prop. XLI.)
3. That doubles of the same magnitude are equal to each other. (Ax. VII.)

Let A B C be the given △, and D E F the given rectilineal ∠.



We have to describe a parallelogram which shall be equal in area to the △ A B C, and have an ∠ equal to the given ∠, D E F.

Bisect the line A B in the point G.

Through the point C draw a straight line, C H K, parallel to the line A B.

From the point G draw a straight line, making with the line A B an ∠ equal to the ∠ D E F.

Let this line meet the line C H K in the point H.

Through B draw a straight line parallel to the line G H, and meeting the line C H produced in the point K.

G B K H is the parallelogram required.

Proof. Join the points C and G by the straight line C G.

The parallelogram G B K H and the △ G B C are upon the same base, G B, and between the same parallels, therefore the parallelogram G B K H is double the △ G B C.

But the △s A G C and G B C are upon equal bases, A G and G B (for A B was bisected at the point G), and have the same altitude. Therefore they are equal to one another, and each is the half of the △ A B C. Or, in other words, the △ A B C is double the △ G B C.

But the parallelogram G B K H is also double the △ G B C.

Therefore, the parallelogram G B K H is equal to the △ A B C.

Moreover, the parallelogram G B K H was constructed with an ∠ equal to the given ∠, D E F.

Consequently, G B K H is the parallelogram required.

It will do equally well if the parallelogram required be constructed on the base A G instead of G B.

If we take any point in the diagonal of a parallelogram (as, *e.g.*, the point E* in the diagonal D B of the parallelogram A B C D), and through it draw lines parallel respectively to the sides of the parallelogram (as H K and F G), it is obvious that the whole parallelogram is divided into four smaller parallelograms. Those two through which the diagonal of the whole parallelogram passes (forming their diagonals) are called *parallelograms about the diagonal* of the whole parallelogram. The two other parallelograms, which, together with the parallelograms about the diagonal, fill up the whole parallelogram, are called the *complements of the parallelograms about the diagonal.*

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* See the diagram in the next proposition.

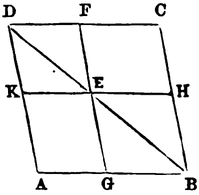
PROPOSITION XLIII.

The complements of any two parallelograms which about the diagonal of a parallelogram are equal to each other.

For proving this proposition we must know,—

1. That a parallelogram is bisected by either of its diagonals. (Prop. XXXIV.)
2. That if equals be added to equals, the sums are equal. (Ax. II.)
3. That if equals be taken from equals, the remainders are equal. (Ax. III.)

Let A B C D be a parallelogram. Let D B be its diagonal, and K E F D and E G B H two parallelograms about the diagonal D B. It has to be shown that the parallelograms K E G A and E H C F (which are the complements of the parallellograms about the diagonal) are equal to each other.



Proof. The △s D C B and D A B are equal to each other, being the two parts into which the parallelogram A B C D is divided by its diagonal D B.

Again, the △s D F E and D K E are equal to each other, being the parts into which the parallelogram D F E K is divided by its diagonal, D E; and, for a similar reason, the △s E H B and E G B are equal to each other.

It follows, therefore, that the sum of the △s D F E and E H B, is equal to the sum of the △s D K E and E G B.

And if these equal sums be taken away from the equal △s DCB and DAB, the remainders will be equal, that is to say, the parallelogram FCHE is equal to the parallelogram KAGE.

PROPOSITION XLIV.

To construct a parallelogram which shall have a given straight line for one of its sides,[*] which shall have an angle equal to a given rectilineal angle, and which shall be equal in area to a given triangle.

For the construction in this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. To produce a straight line to any length. (Post. II.)
3. Through a given point to draw a straight line parallel to a given straight line. (Prop. XXXI.)
4. To construct a parallelogram equal in area to a given △, and having an ∠ equal to a given rectilineal ∠. (Prop. XLIII.)
5. We are also supposed to be able to move a given figure from one position to any other that may be required. (See § 9, p. 8.)

To prove that the construction effects what is required, we must know,—

1. That if two parallel straight lines are intersected by a third, the two interior ∠s are together equal to two right ∠s. (Prop. XXIX.)
2. That if two straight lines are cut by a third, and the two interior ∠s so formed are together less than two right ∠s, those two straight lines will meet if produced. (Ax. XIII.)

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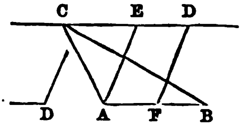
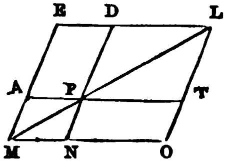
[*] This first condition is usually expressed thus:—To a given straight line to apply a parallelogram which, &c.

3. That if two straight lines cut one another, the vertically opposite ∠s are equal. (Prop. XV.)
4. That the complements of the parallelograms about the diagonal of a parallelogram are equal. (Prop. XLIII.)





Let P T be the given straight line, A B C the given △, and D the given ∠. We have to construct a parallelogram which shall have P T for one of its sides, which shall have an ∠ equal to the ∠ D, and which shall be equal in area to the △ A B C.

Construct a parallelogram, A F D E, equal to the △ A B C, and having an ∠ equal to the ∠ D.

Let this parallelogram be removed to the position A P D E, so as to have one of its sides, A P, in the same straight line with T P, and the ∠ A P D (equal to the given ∠ D) at the point P.

Produce the side E D, and through the point T draw a straight line, T L, parallel to A E, and meeting E D produced in the point L.

Join the points L and P by the straight line L P.

Produce the lines E A and L P. These lines, when produced, will meet; for since L T is parallel to E A, the ∠s A E L and E L T are together equal to two right ∠s. (Prop. XXIX.)

Consequently, the ∠s A E L and E L P are less than two right ∠s, and therefore the lines E A and L P will meet when produced. (Ax. XIII.)

Let them meet in the point M. Through the point M draw the line M O, parallel to A T or E L.

Produce the line D P to meet M O in the point N, and the line L T to meet the line M O in the point O.

By this construction we get a parallelogram, E L O M, subdivided into four other parallelograms, two of which are about the diagonal L M, and the other two are the complements of these.

P T O N is the parallelogram required. For, 1. It has P T for one of its sides. 2. The vertically opposite ∠s, A P D and N P T, formed by the intersection of the straight lines D N and A T, are equal. (Prop. XV. But the ∠ A P D is equal to the ∠ D. Therefore the ∠ N P T is equal to the ∠ D.

3. The parallelogram P T O N is equal to the △ A B C.

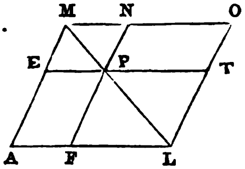
For, since P T O N and A P D E are the complements of the parallelograms about the diagonal L M in the parallelogram E L O M, they are equal. (Prop. XLIII.)

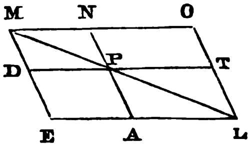
But the parallelogram A P D E was made equal to the △ A B C.

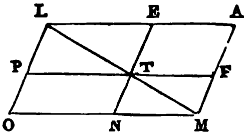
Therefore the parallelogram P T O N is also equal to the △ A B C.

In the construction employed in this proposition it is perfectly indifferent which side of the parallelogram A F D E be placed so as to be in the same straight line with the given line P T; nor does it matter at which end of the line P T it be placed, if the following directions be attended to:—

Place one of the sides of the parallelogram, A F D E, so as to be in the same straight line with P T. Produce the opposite side of the parallelogram in the direction of the line P T. Through that extremity of the line P T at which the parallelogram A F D E was *not* placed, draw a line parallel to either of those sides of the parallelogram which are not in the same straight line as P T, or parallel to it, so as to meet that side of the parallelogram which was produced. Draw that diagonal *of the parallelogram thus formed,* which when produced will pass through that extremity of the line P T, at which the parallelogram A F D E was placed. Produce this diagonal to meet a produced side of the first parallelogram. The mode of completing the figure will then be obvious. The following are some of the various constructions that will result.







PROPOSITION XLV.

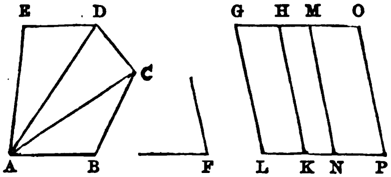
To describe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

For the construction in this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. To describe a parallelogram equal to a given △, and having an ∠ equal to a given rectilineal ∠. (Prop. XLIII.)
3. To construct a parallelogram which shall have a given line for one of its sides, which shall have an ∠ equal to a given rectilineal ∠, and which shall be equal in area to a given △. (Prop. XLIV.)

To prove that the construction accomplishes what is required, we must know,—

1. **That things that are equal to the same are equal to each other. (Ax. I.)**
2. **That if equals be added to equals the sums are equal. (Ax. II.)**
3. **That if two straight lines meet a third at the same point, but on opposite sides of it, and the two adjacent ∠s so formed are together equal to two right ∠s, those two straight lines lie in one and the same straight line. (Prop. XIV.)**
4. **That if two parallel straight lines be intersected by a third straight line, the two interior ∠s are together equal to two right ∠s. (Prop. XXIX.)**
5. **That if two straight lines are both parallel to the same straight line, they are parallel to each other. (Prop. XXX.)**



Let A B C D E be the given rectilineal figure, and F the given ∠. We have to construct a parallelogram equal in area to the figure A B C D E, and having an ∠ equal to the ∠ F.

Divide the rectilineal figure A B C D E into △s by joining the points A and C by the straight line A C, and the points A and D by the straight line A D.*

Construct a parallelogram equal in area to one of these △s (say the △ A B C), and having an ∠ equal to the given ∠, F. (Prop. XLIII.)

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* It does not matter how the figure AB CDE is divided into triangles, nor into how many it is divided.

Let G H K L be the parallelogram so constructed, in which the ∠ G is equal to the given ∠ F.

Construct a parallelogram H M N K, having the line H K for one of its sides, having an ∠ equal to the given ∠ F, and equal in area to the △ A C D; and let it be so placed that the ∠ equal to the given ∠, F, may be adjacent to the ∠ K H G in the parallelogram G H K L, which is not equal to the ∠ F.* (Prop. XLIV.)

Construct a parallelogram M O P N, having the line M N for one of its sides, having an ∠ equal to the given ∠ F, and equal in area to the △ A D E; and let it be so placed that the ∠ equal to the given ∠, F, may be adjacent to the ∠ N M H in the parallelogram H M N K, which is not equal to the given ∠ F.

The figure L G O P will be the parallelogram required.

To show this, it is necessary to prove: 1. That it is a parallelogram. 2. That it is equal in area to the figure A B C D E.

First, then, to prove that it is a parallelogram. For this purpose the first thing that we have to show is, that the lines L K, K N, and N P form one continued straight line, and also that the lines G H, H M, and M O, form one continued straight line.

This is shown as follows:—

Since G L and H K are parallel (being, by construction, opposite sides of a parallelogram), and G H crosses them, the interior ∠s L G H and G H K are together equal to two right ∠s. (Prop. XXIX.)

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* This may always be accomplished by one or other of the various constructions by which the problem proposed in Prop. XLIV. may be solved.

But the ∠ KHM is by construction equal to the ∠ F, and is therefore also equal to the ∠ LGH, which was likewise constructed equal to the ∠ F.

Therefore the sum of the ∠s KHM and KHG is equal to the sum of the ∠s LGH and KHG.

But the sum of the ∠s LGH and KHG is equal to two right ∠s.

Therefore the sum of the ∠s KHM and KHG is also equal to two right ∠s.

We have therefore two right lines, GH and MH, meeting a third right line, KH, at the same point H, and on opposite sides of the line KH, and the two adjacent ∠s so formed are together equal to two right ∠s.

It follows (according to the 14th proposition), that the lines GH and HM form one continued right line.

In a similar manner it may be shown that the ∠s NMO and NMH are together equal to the sum of the ∠s KHM and NMH, and are therefore together equal to two right ∠s, and consequently that the lines HM and MO form one continued right line.

Next, to show that the lines LK, KN, and NP form one continued straight line.

Since the opposite ∠s of parallelograms are equal, it follows that the ∠s LKH, KNM, and NPO are equal respectively to the ∠s LGH, KHM, and NMO, each of which was made equal to the angle F.

Consequently the ∠s LKH, KNM, and NPO are each equal to the ∠ F; and therefore are equal to each other.

Now, because PO and NM are parallel and NP meets them, the interior ∠s NPO and PNM are together equal to two right ∠s.

But the ∠ KNM is equal to the ∠ NPO.

Therefore the sum of the ∠s KNM and PNM is equal to the sum of the ∠s NPO and PNM.

But the sum of the ∠s N P O and P N M is equal to two right ∠s.

Therefore the sum of the ∠s K N M and P N M is also equal to two right ∠s.

We have therefore two right lines K N and P N, meeting a third right line at the same point N, and on opposite sides of the line N M, and making the two adjacent ∠s together equal to two right ∠s.

It follows (according to the 14th proposition), that the lines K N and P N form one continued right line.

In the same way it may be shown that the lines L K and K N form one continued right line.

And since L K is parallel to G H, the whole lines L P and G O are parallel.

Moreover, since K H is parallel to L G, and N M is parallel to K H, it follows that N M is parallel to L G.

And since N M is parallel to L G, and P O is parallel to N M, it follows that P O is parallel to L G.

The figure L P O G is therefore a parallelogram that is, a quadrilateral figure whose opposite sides are parallel).

It remains to show that it is equal in area to the figure A B C D E.

This is evident. For since the three parallelograms, of which the whole parallelogram G O P L is composed, were made equal respectively to the three △s into which the figure A B C D E was divided, it follows that the whole parallelogram L O is equal to the whole figure A B C D E.

PROPOSITION XLVI.

To describe a square upon a given straight line.

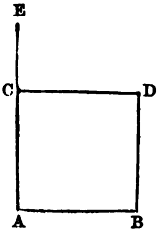
For the construction necessary, we must be able,—

1. From the greater of two given straight lines to cut off a part equal to the less. (Prop. III.)

2. From a given point in a given straight line to draw a perpendicular to that line. (Prop. XI.)
3. Through a given point to draw a straight line which shall be parallel to a given straight line.

To prove that the construction effects what is required we must know,—

1. That things which are equal to the same are equal to each other. (Ax. I.)
2. That the opposite sides and ∠s of a parallelogram are equal. (Prop. XXXIV.)



Let A B be the given straight line. It is required to construct upon it a four-sided figure which shall have all its sides equal, and all its ∠s right ∠s.

From the point A draw the line A E at right ∠s to A B, and cut off from it a part A C, equal to A B.

Through the point C draw a straight line C D parallel to A B.

Through B draw a straight line parallel to A C, and meeting the line C D in the point D.

A B D C is the figure required. For it is obvious from the mode in which it was described, that it is a parallelogram. Consequently, its opposite sides and ∠s are equal. That is to say, the side C D is equal to the side A B, and the side D B to the side A C; the ∠ C D B to the ∠ C A B, and the ∠ A C D to the ∠ A B D.

Hence, since A B and A C are equal to each other, it follows that all the four sides are equal to each other.

Again, since the ∠ C D B is equal to the ∠ C A B, and C A B is a right ∠, the ∠ C D B is also a right ∠.

And since C D and A B are parallel, and C A intersects them, the two interior ∠s D C A and C A B are together equal to two right ∠s.

But C A B is a right ∠. Therefore A C D is also a right ∠.

Therefore, also, the ∠ A B D is a right ∠, since it is equal to the ∠ A C D.

It has thus been shown that A B D C is a four-sided figure, which has all its sides equal, and all its ∠s right ∠s.

PROPOSITION XLVII.

If squares be described upon the three sides of a right-angled triangle, the square which is described upon the hypotenuse (or side subtending the right angle) is equal to the sum of the squares described on the sides which contain the right angle.

For the construction employed in this proposition we must be able,—

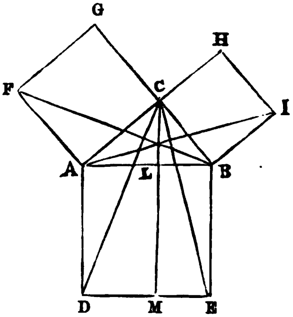
1. To join two given points by a straight line. (Post. 1.)
2. From a given point to draw a straight line which shall be parallel to a given straight line. (Prop. XXXI.)
3. On a given straight line to describe a square. (Prop. XLVI.)

To prove the proposition by the aid of the construction we must know,—

1. That if equals be added to equals, the sums are equal. (Ax. II.)
2. That if two straight lines meet a third at the same point, but on opposite sides of the right line, and make the two adjacent ∠s together equal to two right ∠s, those two right lines form one continued right line. (Prop. XIV.)

3. That if two △s have two sides and the ∠ between them, in the one, equal respectively to two sides and the ∠ between them in the other, those △s are also equal in every other respect. (Prop. IV.)
4. That if a parallelogram and a △ be upon the same base and between the same parallels, the parallelogram is double the △. (Prop. XLI.)

Let A C B be a △ in which the ∠ A C B is a a right ∠.



On the three sides of this △ describe the squares A D E B, A C G F and B I H C.

We have to prove that the square A D E B is equal to the sum of the other two squares A C G F and B I H C.

From the point C draw the line C M parallel to A D (or B E), and meeting the line in D E in the point M, and cutting A B in the point L.

Join the points F and B by the straight line F B; C and D by the straight line C D; C and E by the straight line C E; and A and I by the straight line A I.

The mode in which the proposition is proved is, to show that the square A C G F is equal to the parallelogram A L M D, and the square C B I H to the parallelogram L B E M; so that the sum of the two squares will be equal to the sum of the two rectangles, *i.e.*, to the square A B E D.

To prove that the square A C G F is equal to the parallelogram A L M D, it is shown—1, that the square

A C G M is double of the △ F A B; 2, that the parallelogram A L M D is double of the △ D A C; 3, that these △s are equal to one another; 4, that their doubles are also equal. In a similar manner it is proved, 5, that the square B C H I is double of the △ A B I; 6, that the parallelogram L B E M is double of the △ E C B; 7, that the △s A B I and E C B are equal to one another; and 8, that their doubles are also equal to one another.

1. The square A C G F is double of the △ F A B.

Since G C A is an ∠ of a square, it is a right ∠: the ∠ A C B is also a right ∠. Consequently, we have two straight lines, G C and B C, meeting a third, A C, at the same point C, and on opposite sides, and making the two adjacent ∠s G C A and B C A together equal to two right ∠s.

It follows (according to the 14th proposition), that the two lines G C and C B form one continued straight line. And since the opposite sides of a square are parallel, the line G B is parallel to the line F A.*

Hence it appears that the square (or parallelogram) A C G F and the △ F A B are upon the same base, F A; and between the same parallels, G B and F A.

Therefore the square A C G F is double the △ F A B.

2. The parallelogram D M L A is double the △ D A C; for C M was drawn parallel to A D, and consequently the parallelogram A D M L and the △ C A D are upon the same base, A D; and between the same parallels, A D and C M.

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* Beginners are very apt to omit or misunderstand this part of the proof. They are prone to look upon it as a *part of the construction* that G C B is made one continued straight line, and to forget that it is made up of two separate lines, which must be proved to be one continued right line, before we are entitled to affirm that the whole of G C B is parallel to F A.

Therefore the parallelogram A D M L is double the △ D A C.

3. The △ F A B is equal to the △ D A C.

For the lines F A and A C are equal, being sides of the same square. The lines A B and A D are also equal, being sides of the same square. Moreover, since each of the ∠s F A C and D A B is an ∠ of a square, they are right ∠s, and therefore equal to one another. Add to each of them the ∠ C A B; and the sum of the ∠s F A C and C A B (*i.e.*, the ∠ F A B) will be equal to the sum of the ∠s D A B and C A B (*i.e.*, the ∠ D A C).

Hence the two sides F A and A B and the ∠ F A B in the △ F A B, are equal respectively to the two sides C A and A D and the ∠ C A D in the △ C A D.

Consequently (according to the 4th proposition), these two △s are equal in every respect: and among others, in area.

4. Since the doubles of equals are equal (Ax. VII.), the square F A C G (which is double the △ F A B) is equal to the parallelogram A L M D (which is double the △ C A D).

We have next to show that the square C B I H is equal to the parallelogram L B E M.

The proof of this part of the demonstration is of the same kind as that of the other.

The ∠ H C B is a right ∠, being an ∠ of a square.

The ∠ A C B is also a right ∠.

Therefore, since the two lines A C and H C meet the line B C at the same point and on opposite sides, and make the adjacent ∠s together equal to two right ∠s, those two lines lie in one and the same straight line (Prop. XIV.); and H A is parallel to B I.

Hence it appears that the square B I H C and the △ B I A are upon the same base and between the same parallels.

Therefore the square B C H I is double the △ B I A.

Again, the parallelogram L B E M and the △ E B C are upon the same base, B E; and between the same parallels, B E and C M. Consequently, the parallelogram L B E M is double of the △ E B C.

But the △s ABI and E B C are equal to one another.

For the lines B I and B C are equal, being sides of the same square.

Similarly, the lines B E and B A are equal, being sides of the same square. And since the ∠s C B I and A B E are both right ∠s, and therefore equal to one another, if the ∠ C B A be added to both, the sums are equal. That is to say, the ∠ A B I is equal to the ∠ E B C.

Hence it appears, that the sides A B and B I, and the included ∠ A B I in the one △, are equal respectively to the sides E B and B C, and the included ∠ E B C in the other △.

Therefore, the △ A B I is equal to the △ E B C. (Prop. IV.)

And since the doubles of equals are equal, it follows, that the square C B I H is equal to the parallelogram L B E M.

But if equals be added to equals, the sums are equal (Ax. II.)

Therefore the sum of the squares A C G F and B C H I is equal to the sum of the parallelograms A L M D and L B E M; that is, to the whole square A B E D.

PROPOSITION XLVIII.

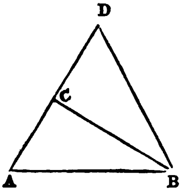
If the square described upon one of the sides of a triangle be equal to the sum of the squares described upon the other two sides, the angle contained by these two sides is a right angle.

For the construction employed in this proposition we must be able,—

1. To join two given points by a straight line. (Post. I.)
2. From a given point in a given right line to draw a right line perpendicular to that line. (Prop. II.)

To prove the proposition by the aid of the construction we must know,—

1. That if the same be added to equals, the sums are equal. (Ax. II.)
2. That things that are equal to the same are equal to each other. (Ax. I.)
3. That if one ∠ of a △ be a right ∠, the square on the side which subtends the right ∠ is equal to the sum of the squares on the sides which contain the right ∠. (Prop. XLVII.)
4. That if two △s have the three sides of the one equal respectively to the three sides of the other, those △s are also equal in every other respect. (Prop. VIII).



Let A C B be a △, such, that, when squares are described upon the three sides of it, the square described on the side A B is equal to the sum of the squares described on the sides A C and C B.

It has to be shown that the ∠ A C B is a right ∠.

From the point C draw a right line perpendicular to the line C B, and from it cut off a part C D, equal to C A.*

Join the points D and B by the right line D B.

Because the lines C A and C D are equal, the squares on them are equal.

If to each of these squares we add the square on C B, the sum of the squares on A C and C B will be equal to the sum of the squares on D C and C B.

But since B C D is a right ∠, the square on the side B D in the △ B C D is equal to the sum of the squares on the sides B C and C D.

Therefore the square on the side B D is also equal to the sum of the squares on A C and C B.

But we know, to start with, that the square on A B is equal to the sum of the squares on A C and C B.

Therefore the square on A B is equal to the square on D B.

Consequently, the line A B must be equal to the line D B.

Hence, the three sides A C, C B, and B A in the △ A C B, are equal respectively to the three sides D C, C B, and B D in the △ D C B.

It follows therefore (according to the 8th proposition) that these △s are also equal in every other respect.

Among these respects is, that the ∠ A C B is equal to the ∠ D C B.

But D C B is a right ∠ (by construction).

Therefore the ∠ A C B is also a right ∠.

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* It would do equally well if a line were drawn perpendicular to C A, and a part cut off, equal to C B.

THE

SECOND BOOK OF EUCLID

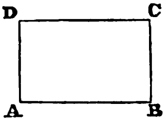
EXPLAINED TO BEGINNERS.

DEFINITIONS.

1. A rectangle is a parallelogram, the angles of which are right angles.

To show that a parallelogram is a rectangle, it is enough to show that one ∠ is a right ∠; because the ∠ opposite to this must be equal to it (I. 34), and any two of the ∠s which are not opposite must be together equal to two right ∠s. (I. 29,)

2. A rectangle is said to be contained by any two of its sides which form one of its ∠s, or (in other words) a rectangle is said to be contained by any two conterminous sides. It is also described as the rectangle under two conterminous sides. Thus the rectangle A B C D is said to be contained by A B and B C, by B C and C D, by A B and. A D, or by A D and D C; or is described as the rectangle *under* A B and B C, &c. A rectangle is also said to be contained by two right lines which are respectively equal to two of its conterminous sides.



3. A square is a rectangle whose conterminous sides are equal.

4. In any parallelogram either of the parallelograms about the diagonal, taken together with the two complements, forms a figure which is called a *gnomon*.

In what follows the abbreviation *rect.* is frequently used for the word *rectangle*, and the symbol □ for the word *square*. Thus "rect. A B, B C," means "the rectangle contained by A B and B C;" "□ on A B" means "the square on A B." Of course, "rects." means "rectangles," and "□s" means "squares."

A rectangle is often named by means of two letters standing at opposite ∠s. Thus the rectangle given in p. 134 may be named as "the rectangle A C," or "the rectangle D B." If lines are named by single letters, "rect. P, Q" means "the rectangle contained by the lines P and Q."

The operations of addition and subtraction are as applicable to magnitudes as to numbers. Two lines or two areas may be added together, or one may be taken from another. Consequently, the signs +, —, and = may be used with relation to magnitudes, as well as with relation to numbers. If A and B are two magnitudes (lines, or angles, or areas), $A + B$ denotes their sum, $A - B$ denotes what is left when B has been taken from A. Such an equation as $A + B = C + D + E$ means that the sum of the two magnitudes, A and B, is equal to the sum of the three magnitudes, A, B, and C. But multiplication is a purely *arithmetical* operation: to speak of *multiplying* a line by a line is absolutely meaningless. Consequently, the signs × and ÷ cannot be used to connect symbols which denote magnitudes. It is utterly wrong and misleading to denote the rectangle contained by two lines, A B and C D by the symbols A B × C D. For a similar reason, it is objectionable to speak of "the square *of* the line A B," because the expression "square *of*" is used with reference to numbers. The "square *of* five" is the product of 5 multiplied by 5. It is the more important to adhere to these distinctions, because (as will be shown further on), there is a very real and close connection between a rectangle and a product which can never be properly understood if the two are confounded from the beginning. The reasoning in several of the propositions in the Second Book of Euclid are rendered much more intelligible by the use of the signs +, —, and =. They will accordingly be occasionally employed.

PROPOSITION I.

If there are two straight lines one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal* to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

For the construction made use of in this proposition we are supposed to be able:—

1. Through a given point in a given straight line to draw a line at right angles to the given line. (I. 11.)
2. From a given right line to cut off a part equal to another given right line. (I. 3.)
3. Through a given point to draw a right line parallel to a given right line. (I. 31.)

We are also supposed to know that:

1. Magnitudes which are equal to the same are equal to each other. (Ax. I.)
2. If a right line intersect two parallel right lines, the exterior angle is equal to the interior opposite angle on the same side of the intersecting line. (I. 29.)
3. The opposite sides of a parallelogram are equal to each other. (I. 34.)

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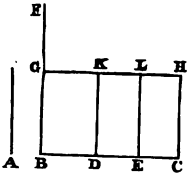
* By *equal* is of course meant *equal in area.*

Let A and B C be two straight lines, of which B C is divided into the parts B D, D E, and E C. We have to show that the rectangle under A and B C is equal to the sum of the rectangles, contained by A and B D, A and D E, and A and E C.



From the point B draw B F at right angles to B C.

From B F cut off B G equal to A. Through the point G draw G H parallel to B C.

Through the points D, E, and C draw D K, E L, and C H parallel to B G, and meeting the line G H in the points K, L, and H.

Since the parallel lines B G and D K are intersected by the right line B C, the ∠ K D E is equal to the ∠ G B C, and is therefore a right ∠.

For a similar reason the ∠ L E C is a right ∠.

Therefore (Def. 1), the parallelograms B H, B K, D L, and E H are all rectangles.

Now it is self-evident that the rectangle B H is equal to the sum of the rectangles B K, D L, and E H.

But the rectangle B H is the rect. under A and B C; for it is contained by B G and B C, and B G is equal to A.

The rectangle B K is the rectangle contained by A and B D, for it is contained by B G and B D, and B G is equal to A.

The rectangle D L is the rectangle contained by A and D E, for it is contained by D K and D E, and D K is equal to B G, which is equal to A.

Also the rectangle E H is the rectangle under A and E C, for it is contained by E L and E C, and E L is equal to B G, which is equal to A.

Therefore the rectangle contained by the lines A and

B C is equal to the sum of the rectangles contained by the undivided line (A) and the several parts (B D, D E, and E C) of the divided line (B C).

The beginner must carefully observe that the gist of the proof does not consist in showing that the rectangle B H is equal to the sum of the rectangles B K, D L, and E H, because that is self-evident; but in showing that the rectangles B H, B K, D L, and E H *answer to the description of them given in the enunciation.*

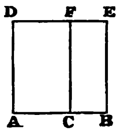
PROPOSITION II.

If a straight line be divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole and each of the parts.

For the construction employed in this proposition we must be able:—

1. On a given straight line to describe a square. (I. 46.)
2. Through a given point to draw a straight line parallel to a given straight line. (I. 31.)

Let A B be a right line divided into two parts at the point C. We have to show that the □ on A B is equal to the sum of rect. A B, A C and rect. A B, B C.



On A B describe the □ A E. Through C draw C F parallel to A D, and meeting D E in the point F.

It is self-evident that the □ A E is equal to the sum of the rectangles A F and C E.

But the □ A E is the □ on A B.

The rectangle A F is the rectangle contained by

A B and A C, for it is contained by A D and A C, and A D and A B are equal, being sides of the same □.

The rect. C E is the rectangle contained by A B and B C, for it is contained by B E and B C, and B E is equal to A B.

Consequently the □ on A B is equal to the sum of the rectangles contained by the whole line (A B), and each of the parts (A C and C B) into which it is divided.

Here again the beginner must observe that the main point in the above proof is, to show that the □ A E, and the rectangles A F and C E answer to the description of them given in the enunciation. We need no proof to show that the □ A E is equal to the sum of the rects. A F and C E.

If in Prop. I. the two lines A and B C were of the same length, and B C were divided into two parts, it is obvious that the second proposition might be considered as only a particular case of the first; for the rect. under A and B C would then be the same as the □ on B C, and the two rects. under A and the parts of the divided line would be the same as the rectangles under the whole line B C and each of its parts.

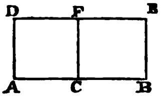
PROPOSITION III.

If a right line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.

The propositions required for the construction employed in this proposition are the same as in the first and second propositions.

Let A B be a right line divided into two parts at C.

We have to prove that the rect. A B, A C is equal to the □ on A C together with the rect. A C, C B.



On A C describe the □ A D F C. Produce D F, and through the point B draw B E, parallel to A D, and meeting D F produced in the point E.

It is self-evident that the rect. A E is equal to the sum of the □ A F and the rect. C E.

But A E is the rectangle contained by A B and AC, for it is contained by A B and A D, and A D and A C are equal, being sides of the same □.

A F is the □ on A C.

C E is the rectangle contained by A C and C B, for it is contained by F C and C B, and F C and A C are equal, being sides of the same □.

Consequently the rectangle under the whole line (A B) and one part (A C) is equal to the square on that part, together with the rectangle contained by the two parts (A C and C B.)

Here again the gist of the proof consists in showing that the rectangle A E, and its parts A F and C E answer to the description given of them in the enunciation.

[The following proposition is made use of in several succeeding propositions.]

PROPOSITION A.

The parallelograms about the diagonal[*] of a square, are themselves squares.

For the proof of this proposition we must know the following,—

1. Magnitudes that are equal to the same are equal to each other.

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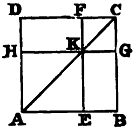
[*] That is parallelograms whose diagonals are parts of the diagonal of the square. See Book I., Prop. 43.

2. The angles at the base of an isosceles triangle are equal. (I. 5.)
3. If two angles of a triangle are equal, the sides opposite those angles are also equal. (I. 6.)
4. If a right line intersect two parallel right lines, the external angle is equal to the internal opposite angle on the same side of the intersecting line, and the two internal angles on the same side are together equal to two right angles. (I. 29.)
5. The opposite sides and angles of a parallelogram are equal. (I. 34.)



Let A B C D be a □, and A C one of its diagonals. Through a point K in A C the lines H G and E F are drawn parallel respectively to A B and A D. H E and F G are parallelograms about the diagonal of the square. We have to prove that they are themselves squares.

First take the parallelogram H E. Because D C and H G are parallel and A C intersects them, therefore the ∠ H K A is equal to the ∠ D C A. But the lines A D and D C are equal because they are sides of the same □; therefore the ∠ D A C is equal to the ∠ D C A.

Consequently the ∠ H A K (*i.e.*, D A C) and the ∠ H K A are equal to each other, for they are both equal to the ∠ D C A.

It follows, therefore (according to I. 6), that the line H A is equal to the line H K.

But H E is a parallelogram; therefore its opposite sides are equal. Consequently H A and K E are equal, and H K and A E are equal. Therefore H E is equilateral, that is, has all its sides equal.

Moreover the ∠s H A E and H K E are opposite ∠s of a parallelogram, and are therefore equal, and H A E is a right ∠, therefore H K E is a right ∠. Also since A B and H G are parallel and A H intersects them, the ∠s H A E and A H K are together equal to two right ∠s; and since H A E is a right ∠, A H K must be a right ∠, and the ∠ K E A, which is opposite and equal to the ∠ A H K, must also be a right ∠.

We have thus shown that the parallelogram H E has all its sides equal, and all its angles right angles; therefore it is a square.

In a similar way it may be shown that the parallelogram F G is a square.

The preceding proposition is usually incorporated in the proof of the Fourth Proposition.

PROPOSITION IV.

If a right line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the parts together with twice the rectangle under the parts.

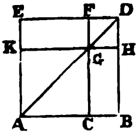
For the construction employed in this proposition we must be able,—

1. To join two points by a right line. (Post. I.)
2. On a given straight line to describe a square. (I. 46.)
3. Through a given point to draw a line parallel to a given straight line. (I. 31.)

For the proof of the proposition we must know,--

1. That the parallelograms about the diagonal of a square are themselves squares. (Prop. A.)
2. That the opposite sides of a parallelogram are equal. (I. 34.)

Let A B be a straight line divided into two parts at C.



On A B describe the □ A B D E. Draw the diagonal A D.

Through C draw C F parallel to A E or B D, intersecting A D in G and meeting E D in F.

Through G draw K G H parallel to A B or E D.

It is self-evident that the whole □ A D is equal to the sum of the four rectangles K C, G B, E G and F H, for they are parts into which it is divided.

But K C and F H are □s (Prop. A.), and K C is the □ on A C, and F H is the □ on C B, for it is the □ on G H which is equal to C B. (I. 34.)

E G is the rectangle contained by A C and C B, for it is contained by K G, which is equal to A C (I. 34) and F G, which is equal to F D, which is equal to C B. (I. 34.)

G B is also the rectangle contained by A C and C B, for it is contained by G C and C B, and G C and A C are equal because they are sides of the same square.

Consequently E G and G B are together equal to twice the rectangle* contained by A C and C B.

We have thus shown that the □ on the whole line (A B) is equal to the sum of the □s on the two parts (A C and C B) together with twice the rectangle under the two parts.

The beginner will observe that the stress of the proof does not consist in showing that the whole figure A D is equal to the sum of the parts into which it is divided, for that is self-evident, but in showing that the whole figure and its parts answer to the description given of them in the enunciation.

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* It would do equally well to point out that the rectangles E G and G B are equal, being complements of the parallelograms K C and F H about the diagonal, and that G B is the rectangle contained by A C and C B; so that E G and G B together are equal to twice the rectangle contained by A C and C B.

PROPOSITION V.

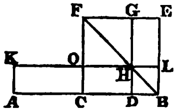
If a right line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

For the construction employed we must be able—

1. On a given straight line to describe a square. (I. 46.)
2. Through a given point to draw a line parallel to a given right line. (I. 31.)

We must also know,—

1. That if equals are added to equals, the sums are equal. (Ax. II.)
2. That the opposite sides of a parallelogram are equal. (I. 34.)
3. That the parallelograms about the diagonal of a square are themselves squares. (Prop. A.)
4. That the complements of the parallelograms about the diagonal of a parallelogram are equal. (I. 43.)
5. That parallelograms upon equal bases and between the same parallels are equal. (I. 36.)



Let the line A B be divided into two equal parts in C, and into two unequal parts in D.

We have to prove that rect. A D, D B, together with the □ on C D, is equal to the □ on C B.

On C B describe the □ C B E F. Draw the diagonal F B.*

Through D draw the right line D G parallel to C F or B E, intersecting F B in H, and F E in G.

Through H draw L K parallel to B A, and through A draw A K parallel to C F or B E, and meeting K L in the point K.

1. Because A C and C B are equal, A O and C L are parallelograms standing on equal bases and between the same parallels; therefore the rectangle A O is equal to the rectangle C L.

2. The rectangles C H and H E are equal, because they are the complements of the parallelograms O G and D L, which are about the diagonal F B.

3. If the equal rectangles C H and H E be added respectively to the equal rectangles A O and C L, the sums will be equal; that is to say, the rectangle A H (which is the sum of A O and C H) will be equal to the gnomon C L G (which is the sum of C L and G L).

4. If to each of these equals the □ O G be added, the sums will be equal; that is to say, the sum of the rect. A H and the □ O G will be equal to the sum of the gnomon C L G and the □ O G.

5. But A H is the rectangle contained by A D and D B, for it is contained by A D and D H, and D H and D B are equal, being sides of the same □; and O G is the □ on C D, for it is the □ on O H which is equal to C D. (I. 34.)

6. Therefore the rect. A D, D B together with the □ on C D is equal to the gnomon C L G together with the □ O G.

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* Let the beginner take particular notice that the diagonal needed is the one which is drawn to *the extremity* of the line A B. The diagonal C E would not do.

7. But the gnomon C L G together with the □ O G is equal to the □ C E, which is the □ on C B.

Therefore the rectangle under the unequal parts, A D and D B, together with the □ on C D, the part between the points of section, is equal to the □ on C B, which is half the line A B.

The steps of this proof may be easily indicated by the use of the following equations, which express what is stated in the preceding paragraphs which bear the same numbers:—

1. Rect. A O = rect. C L.
2. Rect. C H = rect. H E.
3. Therefore rect. A O + rect. C H = rect. C L + rect. H E, or rect. A H = gnomon C L G.
4. Add to both the □ O G, then rect. A H + □ O G = gnomon C L G + □ O G.
5. But A H = rect. A D, D B, and O G = □ on C D.
6. Therefore rect. A D, D B + □ on C D = gnomon C L G + □ O G.
7. But gnomon C L G + □ O G = □ on C B.
8. Therefore rect. A D, D B + □ on C D = □ on C B.

If A C and C D be regarded as two separate lines, it is clear that A D is the sum of the two, and D B the difference between them, for D B is what is left when C D is taken from C B, and C B is equal to A C. Therefore A H is the rectangle under the sum and the difference of the two lines A C and C D. O G is the □ on the smaller of the two lines; and C E is the □ on the larger, for it is the □ on C B, which is equal to A C.

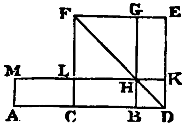
Consequently what is proved in this proposition may be thus stated: "The rectangle contained by the sum and the difference of two unequal lines, together with the □ on the smaller, is equal to the □ on the larger;" or thus: "The difference between the □s on two unequal lines (as

A C and C D) is equal to the rectangle contained by their sum and their difference.[*]" This is incomparably the most useful form in which the proposition can be stated.

PROPOSITION VI.

If a right line be bisected and produced to any length, the rectangle under the whole line thus produced and the produced part, together with the square on half the line, is equal to the square on the line made up of the half line and the produced part.

The propositions required for the construction and proof in this proposition are the same as in the last.



Let A B be a right line bisected in C, and produced to D. We have to prove that the rect. A D, D B, together with the □ on C B, is equal to the □ on C D.

On C D describe the □ C D E F.

Draw the diagonal F D (see the note on the last proposition.)

Through B draw B G parallel to C F or D E, cutting F D in H and F E in G.

Through H draw K H M parallel to A D, and through A draw A M parallel to D E, and meeting K M in the point M.

1. The rectangles A L and C H are on equal bases

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[*] The beginner will remember that the difference between two magnitudes is what is added to the smaller to make a quantity equal to the larger; and the rectangle A H is added to the □ O G to make a quantity equal to the □ C E.

(A C and C B) and between the same parallels, therefore they are equal to each other.

2. The rectangles C H and H E are the complements of the parallelograms L G and B K, which are about the diagonal F D; therefore they are equal to each other.

3. It follows therefore that the rect. A L is equal to the rect. H E.

4. If to each of these equal rectangles we add the rect. C K, the sums will be equal, that is, the rect. A K will be equal to the gnomon C K G.

5. If now to each of these equals we add the □ L G, the rect. A K together with the □ L G will be equal to the gnomon C K G together with the □ L G, that is, to the whole □ C E.

6. But A K is the rectangle contained by A D and D B, for it is contained by A D and D K, and D K and D B are equal, being sides of the same square.

7. L G is the □ on C B, for it is the □ on L H, and L H is equal to C B (I. 34); and C E is the □ on C D.

8. We have thus shown that the rectangle contained by the whole produced line (A D) and the part produced (B D), together with the □ on half the line (C B), is equal to the □ on C D, which is the line made up of the half C B and the produced part B D.

The steps of the preceding proof are exhibited in the following equations:—

1. Rect. A L = rect. C H.
2. Rect. C H = rect. H E.
3. Therefore rect. A L = rect. H E.
4. Add rect. C K to both, then
   Rect. A L + rect. C K = rect. H E + rect. C K.
   Or rect. A K = gnomon C K G.
5. Add to both the □ L G, then
   Rect. A K + □ L G = gnomon C K G + □ L G.
   or rect. A K + □ L G = □ C E.

6. But rect. A K = rect. A D, D B, and
7. □ L G = □ on C B, and □ C E is □ on C D.
8. Therefore rect. A D, D B + □ on C B = □ on C D.

If we regard A C and C D as two separate lines, it is clear that A D is their sum, and B D their difference, for B D is what is left when C B (or A C, which is equal to C B) is taken from C D. C E is the □ on C D, and L G is the □ on A C, for it is the □ on L H, which is equal to C B which is equal to A C. The difference between the □s C E and L G is the gnomon C K G, which is equal to the rectangle A K.

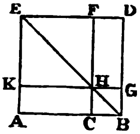
Consequently this proposition may be expressed in exactly the same form as the last, namely: The difference between the □s on two lines (C D and C A) is the rectangle contained by their sum (A D) and their difference (B D.)

The segments of a right line made by a point in it are the distances of the point from the two extremities of the line. This definition of *segments* may be extended to the case where the point is not in the line itself, but in the line *produced;* so that in Prop. VI. D A and D B are termed *segments* of the line A B. The fifth and sixth propositions may now be combined into one in this way: The rectangle contained by the segments of a line formed by a point (either in the line or in the line produced) is the difference between the □ on half the line and the □ on the distance of the point of section from the middle of the line. In the fifth proposition the former of these two □s is the larger, in the sixth, the latter.

PROPOSITION VII.

If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are together equal to twice the rectangle contained by the whole and that part together with the square on the other part.

The propositions required for the construction and proof in this proposition are the same as in the fourth, together with Prop. A (p. 140.)



Let A B be a straight line divided into two parts at the point C. We have to show that the □ on A B together with the □ on one part (say C B) is equal to twice the rect. A B, B C together with the □ on A C.

On A B describe the □ A B D E.

Draw the diagonal E B.

Through C draw C F parallel to A E or B D, intersecting E B in H, and E D in F.

Through H draw G H K parallel to A B, meeting B D in G and A E in K.

1. K F and C G are □s (Prop. A): and K F is the □ on A C, for it is the □ on K H, and K H is equal to A C (I. 34); and C G is the □ on C B.

2. It is self-evident that the □ A D is equal to the rectangles A G and H D, together with the □ K F.

3. To each of these equals add the □ C G, and the □s A D and C G together will be equal to the sum of the □ K F, the rect. A G, the rect. H D, and the □ C G.

4. But the rect. H D together with the □ C G is equal to the rect. C D.

5. Therefore the □ A D together with the □ C G is equal to the □ K F, the rect. A G and the rect. C D taken together.

6. But the rectangles A H and H D are equal, being complements of the parallelograms about the diagonal of A D. Therefore the sum of A H and C G (*i. e.*, the rectangle A G) is equal to the sum of H D and C G (*i. e.*, the rectangle C D.)

7. Therefore the sum of A G and C D is the same as

twice the rectangle A G, which is the rectangle under A B and B C, for B G is equal to B C.

8. It follows therefore that the □ on A B, together with the □ on C B is equal to twice the rect. A B, B C, together with the □ K F, which is the □ on A C.

The beginner must not fail to notice that C G is the piece by which the rectangles A G and C D overlap each other.

The leading steps of the above demonstration may be clearly shown by the aid of equations:—

2. □ A D = □ K F + rect. A G, + rect. H D.

3. Add □ C G to both sides; then □ A D + □ C G = □ K F + rect. A G + rect. H D + □ C G.

4. But rect. H D + □ C G = rect. C D.

5. Therefore □ A D + □ C G = □ K F + rect. A G + rect. C D.

6. Now rect. A H = rect. H D.

Therefore rect. A H + □ C G = rect. H D + □ C G, that is, rect. A G = rect. C D.

7. Therefore rect. A G + rect. C D = 2ce rect. A G, or rect. A G + rect. C D = 2ce rect. A B, B C.

8. □ A D + □ C G = □ K F + 2ce rect. A G, or □ on A B + □ on B C = 2ce rect. A B, B C + □ on A C.

In this proposition again the learner will observe that the bulk of the proof consists in showing that the various squares and rectangles compared with each other answer to the description given of them in the enunciation.

Prop. VIII. is usually passed over, as it is troublesome and useless.

PROPOSITION IX.

If a straight line be divided into two equal and also into two unequal parts, the squares on the two unequal parts are together equal to twice the square on half the line together with twice the square on the line between the points of section.

In the construction employed in this proposition we must be able,—

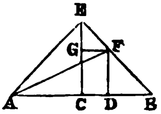
1. To join two given points by a straight line. (Post. I.)
2. From the greater of two given straight lines to cut off a part equal to the less.
3. To draw a straight line perpendicular to a given straight line from a given point in the same. (I. 11.)
4. Through a given point to draw a straight line parallel to a given straight line. (I. 31.)

To prove the proposition by the aid of the figure constructed, we must know,—

1. That the ∠s at the base of an isosceles triangle are equal. (I. 5.)
2. That if two ∠s of a triangle are equal, the sides opposite them are also equal. (I. 6.)
3. That the three ∠s of a triangle are together equal to two right ∠s. (I. 32.)
4. That if a straight line intersect two parallel straight lines, the exterior ∠ is equal to the interior opposite ∠ on the same side of the intersecting line, and the two interior ∠s on the same side are together equal to two right ∠s. (I. 29.)
5. That the opposite sides of a parallelogram are equal. (I. 34.)

6. That in a right-angled triangle the □ on the hypotenuse is equal to the sum of the □s on the other two sides. (I. 47.)

Let A B be a straight line, bisected at C, and divided into two unequal parts at D. We have to prove that the □ on A D together with the □ on D B is equal to twice the □ on A C together with twice the □ on C D.



From C draw C E at right ∠s to A B, and cut off C E equal to C A or C B.

Join A and E, and E and B.

Through D draw D F parallel to C E, and meeting E B in the point F.

Through F draw F G parallel to A B, meeting C E in G.

Join A and F.

The first portion of the proof consists in showing with regard to this figure:

1. That A E F is a right ∠.
2. That A D F is a right ∠.
3. That G E is equal to G F.
4. That D F is equal to D B.

1. *To prove that* A E F *is a right* ∠ :—

C A is equal to C E, therefore the ∠ C E A is equal to the ∠ C A E. (I. 5.)

The three ∠s of the △ A C E are together equal to two right ∠s (I. 32) and A C E is a right ∠, therefore C A E and C E A are together equal to one right ∠.

Consequently, as the ∠s C A E and C E A are equal to each other, C E A is half a right ∠.

In a similar way it may be shown that C E B is half a right ∠.

It follows, therefore, that A E B (that is, A E F) is a right ∠.

2. *To prove that* ADF *is a right* ∠ :—

The parallels EC and FD are intersected by CD; therefore the two ∠s GCD and CDF are together equal to two right ∠s. GCD is a right ∠, therefore FDC is a right ∠.

3. *To prove that* GE *is equal to* GF :—

The straight line EC intersects the parallel straight lines GF and CD, therefore the ∠ EGF is equal to the ∠ GCD. (I. 29.)

But GCD is a right ∠, therefore EGF is a right ∠.

In the △ EGF the three ∠s are together equal to two right ∠s; EGF is a right ∠, and GEF is half a right ∠; consequently, the ∠ GFE is half a right ∠, and is equal to the ∠ GEF.

It follows, therefore (I. 6), that GE is equal to GF.

4. *To prove that* DF *is equal to* DB :—

The three ∠s of the △ ECB are together equal to two right ∠s; ECB is a right ∠, and CEB is half a right ∠; therefore CBE (that is, DBF) is half a right ∠.

The straight line CB intersects the parallel lines FD and GC; therefore the ∠ FDB is equal to the ∠ GCB.

But GCB is a right ∠, therefore FDB is a right ∠.

The three ∠s of the △ FDB are together equal to two right ∠s. (I. 32.) FDB is a right ∠, and DBF is half a right ∠; consequently, the ∠ DFB is half a right ∠, and is equal to the ∠ DBF.

Because the ∠ DFB is equal to the ∠ DBF, DF is equal to DB.

[The remainder of the proof depends upon the fact that AF is the hypotenuse of *two* right-angled △s, namely, AEF and ADF; so that the sum of the □s on AD and DF admits of being compared with the sum of the □s on AE and EF. If this point is clearly kept in mind, the course of the proof will be easily remembered. It will be proved (5) that the □ on A F is equal to the sum of the □s on A D and D F, or A D and D B; (6) that the □ on A F being equal to the sum of the □s on A E and E F, is also equal to twice the □ on A C, together with twice the □ on C D; whence it follows (7) that the sum of the □s on A D and D B is equal to twice the □ on A C together with twice the □ on C D.]

5. *To prove that the* □ *on* A F *is equal to the sum of the* □*s on* A D *and* D B :—

A D F is a right ∠, therefore the □ on A F is equal to the sum of the □s on A D and D F.

But D F is equal to D B, therefore the □ on D F is equal to the □ on D B. Consequently, the sum of the □s on A D and D F is equal to the sum of the □s on A D and D B; and therefore the □ on A F is equal to the sum of the □s on A D and D B.

6. *To prove that the* □ *on* A F *is equal to twice the* □ *on* A C, *together with twice the* □ *on* C D :—

The ∠ A E F is a right ∠, therefore in the △ A E F the □ on A F is equal to the sum of the □s on A E and E F. (I. 47.)

But in the △ A C E the ∠ A C E is a right ∠, therefore the □ on A E is equal to the sum of the □s on A C and C E.

But A C is equal to C E (by construction), therefore the □ on A C is equal to the □ on C E. Consequently, the sum of the □s on A C and C E is equal to twice the □ on A C.

Therefore the □ on A·E is equal to twice the □ on A C.

Again, in the △ E G F the ∠ E G F is a right ∠, therefore the □ on E F is equal to the sum of the □s on E G and G F, and therefore to twice the □ on G F, because G F is equal to G E.

But G F and C D are equal, being opposite sides of the parallelogram G D (I. 34); therefore twice the

□ on G F is equal to twice the □ on C D; and, consequently, the □ on E F is equal to twice the □ on C D.

It follows, therefore, that the □ on A F is equal to twice the □ on A C, together with twice the □ on C D.

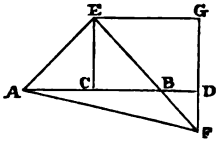
7. It has been proved that the □ on A F is equal to the sum of the □s on A D and D B, and also to twice the □ on A C together with twice the □ on C D.

It follows, therefore, that the sum of the □s on A D and D B is equal to twice the □ on A C together with twice the □ on C D.

PROPOSITION X.

If a straight line be bisected and produced to any point, the square on the whole line thus formed together with the square on the produced part, is equal to twice the square on half the line together with twice the square on the line made up of the half and the part produced.

The propositions required for the construction and proof in this proposition are the same as in the last.



Let A B be a right line bisected in C, and produced to D. We have to prove that the □ on A D together with the □ on D B, is equal to twice the □ on A C, together with twice the □ on C D.

From C draw C E perpendicular to A B, and cut off C E equal to A C or C B.

Join A and E, and B and E.

Through E draw E G parallel to A B, and through D draw D G parallel to E C, meeting E G in G.

Because E C and G D are parallel and E G intersects them, the ∠s C E G and E G D are together equal to two right ∠s. (I. 29.) Therefore the ∠s B E G and E G D are together less than two right ∠s; and consequently the lines E B and G D will meet if produced. Produce them, and let them meet in the point F.

Join A and F.

The proof of this proposition is step for step, and almost word for word, the same as that of the last. It is proved,—

1. That the ∠ A E F is a right ∠.
2. That A D F is a right ∠.
3. That G E is equal to G F.
4. That D F is equal to D B.
5. That the □ on A F is equal to the sum of the □s on A D and D B.
6. That the □ on A F is equal to twice the □ on A C, together with twice the □ on C D.
7. Whence it follows, lastly, that the sum of the □s on A D and D B is equal to twice the □ on A C together with twice the □ on C D.

1. *To prove that* A E F *is a right* ∠ :

In the △ A C E, A C and C E are equal, therefore the ∠s C A E and C E A are equal. And because the three ∠s of the △ A C E are together equal to two right ∠s, and A C E is a right ∠, it follows that C A E and C E A are together equal to a right ∠, and therefore C E A is half a right ∠.

In a similar manner it may be proved that the ∠ C E B is half a right ∠.

Consequently, A E F is a right ∠.

2. *To prove that* A D F *is a right* ∠ :

In the parallelogram C G the opposite ∠s E C D and E G D are equal (I. 34); therefore E G D is a right ∠.

The line F G intersects the parallels E G and A D, therefore the ∠ A D F is equal to the ∠ E G F. Consequently A D F is a right ∠.

3. *To prove that* G E *is equal to* G F.

Since E F intersects the parallels E C and G F, the alternate ∠s C E F and E F G are equal. But C E F is half a right ∠, therefore E F G is half a right ∠. And since the three ∠s of the △ E F G are together equal to two right ∠s, and E G F is a right ∠, and E F G is half a right ∠, it follows that G E F is half a right ∠, and is equal to the ∠ G F E.

Consequently (I. 6) G E is equal to G F.

4. *To prove that* D F *is equal to* D B.

In the △ B D F, the ∠ B D F is a right ∠, D F B is half a right ∠, therefore D B F is half a right ∠, and is equal to D F B.

Therefore D F is equal to D B.

5. *To prove that the* □ *on* A F *is equal to the sum of the* □*s on* A D *and* D B.

A D F is a right ∠, therefore in the △ A D F the □ on A F is equal to the sum of the □s on A D and D F.

But D F is equal to D B; therefore the □ on D F is equal to the □ on D B.

Therefore the □ on A F is equal to the sum of the □s on A D and D B.

6. *To prove that the* □ *on* A F *is also equal to twice the* □ *on* A C, *together with twice the* □ *on* C D.

In the △ A E F, the ∠ A E F is a right ∠; therefore the □ on A F is equal to the sum of the □s on A E and E F.

In the △ A C E the ∠ A C E is a right ∠; therefore the □ on A E is equal to the sum of the □s on A C and C E, and is therefore equal to twice the □ on A C, since A C and C E are equal.

Again, in the △ E G F, the ∠ E G F is a right ∠, therefore the □ on E F is equal to the sum of the □s on E G and G F, and is therefore equal to twice the □ on E G, since E G is equal to G F.

But E G and C D are equal, because they are opposite sides of the parallelogram C G. Therefore twice the □ on E G is equal to twice the □ on C D.

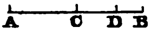
It follows therefore that the □ on A F is equal to twice the □ on A C together with twice the □ on C D.

7. It has thus been proved that the □ on A F is equal to the sum of the □s on A D and D B, and also to twice the □ on A C, together with twice the □ on C D.

It follows therefore that the sum of the □s on A D and D B is equal to twice the □ on A C, together with twice the □ on C D.

It is an excellent mental exercise to master the elaborate proof given by Euclid of the ninth and tenth propositions; but the result might have been arrived at in a much shorter way. The following is one of the proofs that have been given:

Let A B be bisected in C, and divided unequally in D.



It has been proved in the fourth proposition that □ on A D = □ on A C + □ on C D + 2^{ce} rect. A C, C D.

But A C = B C.

Substitute B C for A C in this equation, and we get,—

□ on A D = □ on B C + □ on C D + 2^{ce} rect. B C, C D.

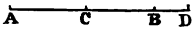
Add to both sides the □ on D B, then □ on A D + □ on D B = □ on B C + □ on C D + 2^{ce} rect. B C, C D + □ on D B.

But 2^{ce} rect. B C, C D + □ on D B = □ on B C + □ on C D. (II. 7.)

Substitute the second side of this equation for the first in the previous equation, and we get,—

□ on A D + □ on D B = □ on B C + □ on C D + □ on B C + □ on C D; *i.e.*, □ on A D + □ on D B = 2^{ce} □ on B C + 2^{ce} □ on C D.

For the 10th proposition, let A B be bisected in C and produced to D.



□ on A D = □ on A C + □ on C D + 2^{ce} rect. A C, C D.

But C B = A C.

Substitute C B for A C in the previous equation, and we get,—

□ on AD = □ on CB + □ on CD + 2^{ce} rect. CB, C D.

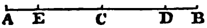
Add the □ on D B to both sides, then □ on A D + □ on D B = □ on C B + □ on C D + 2^{ce} rect. C B, C D + □ on D B.

But 2^{ce} rect. C B, C D + □ on D B = □ on C D + □ on C B. (II. 7.)

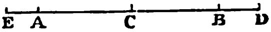
Substitute the second side of this equation for the first in the previous equation, and we get,—

□ on A D + □ on D B = □ on CB + □ on C D + □ on C B + □ on C D; *i.e.*, □ on A D + □ on D B = 2^{ce} □ on C B + 2^{ce} □ on C D.

If in Prop. IX. A D and D B be considered as two separate lines, it is clear that A B is their sum, and A C is half their sum. C D is what is left when D B is taken from C B, therefore it is equal to what would be left if D B were taken from A C. Therefore, if A E be cut off from A C equal to D B, E D will be bisected in C. But E D is the difference between A D and A E or D B, therefore C D is half the difference between A D and D B. It follows, therefore, that the 9th proposition may be thus expressed :—"*The sum of the squares on two lines is equal to twice the square on half their sum, together with twice the square on half their difference.*"



The very same relation is proved in the 10th proposition. Let A D and D B be regarded as separate lines. Let B A be produced, and A E cut off equal to B D. It is then evident that E D is the sum of A D and D B, and consequently C D is half their sum. A B is the difference between A D and D B, and therefore A C is half their difference.



PROPOSITION XI.

To divide a given finite right line so that the rectangle contained by the whole line and one segment shall be equal to the square on the other segment.

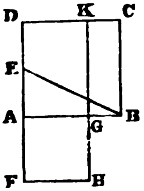
For the construction required we must be able,—

1. To produce a straight line to any length.
2. From the greater of two given straight lines to cut off a part equal to the less. (I. 3.)
3. To draw a straght line parallel to a given straight line. (I. 31.)
4. On a given straight line to describe a square. (I. 46.)

For the proof we must know,—

1. That if a line be bisected and produced, the rectangle contained by the whole line and the produced part, together with the □ on half the line, is equal to the □ on the line made up of the half and the produced part. (II. 6.)
2. In a right-angled △, the □ on the hypotenuse is equal to the sum of the □s on the other two sides.

Let A B be the given line. On A B describe the □ A B C D. Bisect D A in E, and join E and B.



Produce E A, and cut off E F equal to E B; and from A B cut off A G, equal to A F.

G is the point of section required.

Now complete the square A H by drawing G H parallel to A F, and F H parallel to A G, and produce H G to meet D C in K.

D A is bisected in E, and produced to F; consequently (II. 6), the rect. D F, F A (that is, the rectangle D H), together with the □ on E A, is equal to the □ on E F.

But E F is equal to E B, therefore the □ on E F is equal to the □ on E B.

Therefore the rect. D H, together with the □ on E A, is equal to the □ on E B.

But E A B is a right-angled △, therefore the □ on E B is equal to the sum of the □s on E A and A B; and consequently the rect. D H, together with the □ on E A, is equal to the sum of the □s on E A and A B.

Take away the □ on E A from both these equals, and we have left the rect. D H equal to the square on A B, that is, the square D B.

From these take away the rect. D G, which is common to both, and we have left the □ A H, equal to the rect. K B.

But A H is the □ on A G, and K B is the rect. A B, B G, for it is contained by C B and B G, and C B is equal to A B.

Therefore the line A B has been divided so that the rect. A B, B G is equal to the □ on A G.

A line thus divided is said to be divided in extreme and mean ratio.

It is evident from the course of the proof that the line D F is divided at A so that the rectangle under D F and F A is equal to the □ on D A. If from A D a part were cut off equal to A G, A D would be divided exactly as A B is divided. But A D is the larger of the two parts into which D F is divided at A, and A F is the smaller. It follows, therefore, that if a line be divided in extreme and mean ratio, and the smaller segment be measured off upon the larger, this larger segment will itself be divided in a similar manner. It is clear, also, that this process admits of being continued *ad infinitum*.

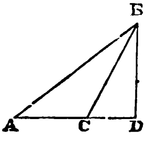
PROPOSITION XII.

In an obtuse-angled triangle the square on the side subtending the obtuse angle exceeds the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by either of them and the distance between a perpendicular drawn to that side produced from the opposite angle, and the point of the obtuse angle.

For the proof, we must know,—

1. That in a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.
2. That if a line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the parts, together with twice the rectangle contained by the parts. (II. 4.)

Let A B C be a △ having an obtuse ∠, A C B. Let B D be drawn perpendicular to A C produced.



We have to prove that the □ on A B exceeds the sum of the □s on A C and C B by twice the rect. A C, C D.

Since A D is divided into two parts at C, the □ on A D is equal to the sum of the □s on A C and C D, together with twice the rect. A C, C D. (II. 4.)

To each of these equals add the square on B D.

Then the □ on A D together with the □ on B D is equal to the sum of the □s on A C, C D, and B D, together with twice the rect. A C, C D.

But the ∠ at D is a right ∠, therefore the □ on A B is equal to the sum of the □s on A D and D B, and the □ on C B is equal to the sum of the □s on C D and D B.

It follows, therefore, that the □ on A B is equal to the sum of the □s on A C and C B, together with twice the rect. A C, C D; that is, the □ on A B exceeds the sum of the □s on A C and C B by twice the rect. A C, C D.

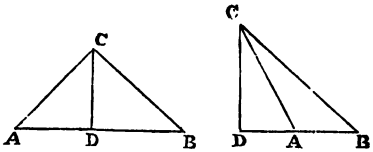
PROPOSITION XIII.

In any triangle the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle by twice the rectangle contained by either of them and the distance between a line drawn perpendicular to this side (or its production) from the opposite angle, and the point of the acute angle.

For the proof of this proposition we must know,—

1. That in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.
2. That if a line be divided into any two parts, the sum of the squares on the whole line and one part is equal to twice the rectangle contained by the whole and that part, together with the square on the other part. (II. 7.)

Let A B C be a △ in which the ∠ A B C is an acute angle, and let C D be drawn perpendicular to B A, or B A produced.



We have to prove that the □ on A C is less than the sum of the □s on A B and B C by twice the rect. A B, B D.

Since A B (in the first figure) and D B (in the second) are divided into two parts, the sum of the □s on A B and B D (in either case) is equal to twice the rect. A B, B D, together with the □ on A D.* (II. 7.)

To each of these equals add the □ on C D; then the sum of the □s on A B, B D, and C D, is equal to twice the rect. A B, B D, together with the □s on A D and D C.

But since C D B is a right angle, the □ on C B is equal to the sum of the □s on C D and D B, and the □ on C A is equal to the sum of the □s on C D and D A.

Consequently (substituting equals for equals), the sum of the □s on A B and B C is equal to twice the rect. A B, B D, together with the □ on C A; that is, the square on C A is less than the sum of the squares on C B and A B by twice the rect. A B, B D.

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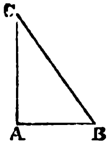
* The beginner will notice that in applying Prop. 7 to the second figure it must be read thus:—"If a line be divided into two parts, the sum of the □s on one part and the whole is equal to twice the rectangle contained by that part and the whole, together with the □ on the other part."



If the perpendicular drawn from C to the opposite side coincide with one side (C A) of the △, then it is obvious that the □ on C A is less than the □ on C B by the □ on A B (I. 47); consequently, the □ on C A is less than the sum of the □s on A B and B C by twice the □ on A B, *i.e.*, by twice the rectangle contained by A B and B A.

PROPOSITION XIV.

To describe a square that shall be equal to a given rectilineal figure.

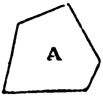
For the construction we must be able,—

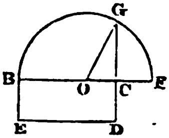
1. To produce a given finite right line. (Post. 2.)
2. To describe a circle with a given point or centre, and at a given distance from that centre. (Post. III.)
3. From the greater of two given right lines to cut off a part equal to the less. (I. 3.)
4. To describe a rectangular parallelogram equal to a given rectilineal figure. (I. 45.)
5. To bisect a given finite right line. (I. 10.)

For the proof we must know (besides the axioms),—

1. That in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. (I. 47.)
2. That if a line be divided into two equal, and also into two unequal parts, the rectangle contained by the unequal parts together with the square on the part between the points of section is equal to the square on half the line. (II. 5.)

Let A be the given rectilineal figure.* Describe the rectangle B D equal to A. (I. 45.) Produce B C, and cut off C F equal to C D. Bisect B F in O, and with O as centre, at the distance O B or O F describe a circle, B G F.





Produce D C to meet the circle in G. The □ on C G will be the □ required.

Now join O and G.

Because B F is bisected in O and divided unequally in C, the rect. B C, C F, together with the □ on O C, is equal to the □ on O F, or to the □ on O G, which is equal to O F. (II. 5.)

But since O C G is a right-angled △, the □ on O G is equal to the sum of the □s on O C and C G.

Therefore the rect. B C, C F, together with the □ on O C, is equal to the sum of the □s on O C and C G.

Take away the □ on O C from both these equal sums, and it follows that the rect. B C, C F is equal to the □ on C G.

But the rect. B C, C F is the same as the rect. B D, for C F is equal to C D.

Therefore the rectangle B D is equal to the □ on C G.

But B D is equal to the figure A; therefore the □ on C G is equal to the figure A.

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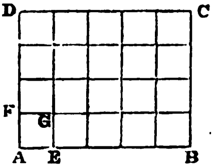
* The introduction of this rectilineal figure is a perfectly needless proceeding. The proposition is (essentially), "To construct a square equal to a given rectangle."

On the numerical relation between the area of rectangles and the length of their sides.



If A B C D be a rectangle, the sides of which contain respectively 4 and 5 times a certain unit of length (A E), and the sides be divided into 4 and 5 of this unit, and lines be drawn through the points of division parallel to the sides of the rectangle, it is obvious that the whole rectangle will be divided into 4 × 5 equal squares, each of which is the square on A E.

It will be easily seen that the same relation holds good for any other numbers. That is (using algebraical notation), if the sides that contain a rectangle are divisible exactly into *a* and *b* linear units respectively (*a* and *b* being whole numbers), the rectangle will contain *ab* times the square on that unit.

The same relation holds good if *a* and *b* are fractional numbers. Suppose the sides of a rectangle contained $\frac{3}{7}$ and $\frac{23}{5}$ of the linear unit. If we divide the unit into 35 equal parts, the sides will contain respectively 3 × 5, or 15, and 23 × 7, or 161 of these sub-units, and the rectangle will contain (3 × 5) × (23 × 7), or 15 × 161 times the square on this sub-unit. But since it takes 35 × 35 times the square on the sub-unit of length to make up the square on the original unit, it follows that the rectangle contains $\frac{15 \times 161}{35 \times 35}$ $\left(i.e. \frac{(3 \times 5) \times (23 \times 7)}{(5 \times 7) \times (5 \times 7)}\right)$ times the square on the original unit. But $\frac{(3 \times 5) \times (23 \times 7)}{(5 \times 7) \times (5 \times 7)} = \frac{3 \times 23}{7 \times 5}$. Therefore, here again the whole or fractional number of times that the rectangle contains the square on the linear unit is obtained by multiplying the fractional number that denotes how much of the linear unit is contained in one side, by the fractional number that denotes how much of the linear unit is contained in the other side.

As it is obvious that similar reasoning applies to every case, it may be stated generally, that if the sides of a rectangle contain $\frac{m}{n}$ and $\frac{p}{q}$ of a certain linear unit, the rectangle contains $\frac{mp}{nq}$ of the square on the linear unit. Or, if a and b (whole or fractional) denote the lengths of two sides of a rectangle in terms of some linear unit, ab denotes the area of the rectangle in terms of the square on that unit. If the sides are equal, and each contains a linear units, the area contains a^2 square units.

It is sometimes stated loosely that the area of a rectangle is the product of the sides that contain it. This, taken literally, has no signification whatever. It is absurd to talk of multiplying a *line* by a *line*. Before any sense can be attached to the words, they must be translated into the proposition previously stated. The sign × can only be placed between symbols that denote numbers. If ever the expression $AB \times CD$ be employed, A B and C D being two lines, it must be understood that, for the purpose in hand, A B and C D are *numbers*, denoting the *number* of linear units contained in A B and C D respectively.

The following propositions are obviously true:—"Those magnitudes are equal which are represented (in terms of the same unit) by equal numbers, or numerical expressions;" and the *quantity* of space denoted by a number is independent of the *form* of the space; m square units make the same quantity of space, whatever the way in which they are arranged or grouped together. Hence (for example), if $a + b + c - d = m$ (that is, if the final result of the operation indicated on the first side be m), the quantity of space filled by m square units will be equal to what we get if to the space occupied by a square units we add successively the spaces occupied by b and c square units, and then take away the space occupied by d square units. Thus, if $a = m + n$, the area denoted by ab is equal to the area denoted by $b\ (m + n)$, because the numbers ab and $b\ (m + n)$ are equal. But $b\ (m + n) = bm + bn$. Consequently, the quantity of space represented by $b\ (m + n)$ is equal to the quantity of space represented by $bm + bn$; and therefore the space represented by ab is equal to the quantity represented by $bm + bn$. But bm denotes the area of a rectangle whose sides contain respectively b and m units; and bn denotes the area of a rectangle whose sides contain respectively b and n units. Hence, the area of a rectangle, whose sides contain respectively a and b units, is the same as the sum of two rectangles whose sides contain respectively b and m and b and n linear units, when $a = m + n$.

The application of these principles enables us to exhibit the propositions of the Second Book of Euclid in a very easy form.

PROPOSITION I.

Let two lines, A and B, contain a and b linear units, and let $b = m + n + p$; that is, let B be divided into parts containing respectively m, n, and p units. The rect. A, B contains ab square units.

But $ab = a\ (m + n + p)$ and $a\ (m + n + p) = am + an + ap$ $\therefore ab = am + an + ap$; and therefore the area denoted by ab is equal to that denoted by $am + an + ap$.

Now the area denoted by $am + an + ap$ is the same as that of the sum of three rectangles whose sides contain respectively a and m, a and n, and a and p linear units.

Therefore, the rectangle contained by A and B is equal to the sum of the rectangles contained by B and the several parts of the divided line.

Propositions II. and III. may be proved in a similar manner. It was shown (p. 139), that they are only particular cases of Prop. I.

PROPOSITION IV. (See p. 142.)

Let A B contain a linear units, and let $a = b + c$. The square on A B contains a^2 square units. But since $a = b + c$, $a^2 = (b + c)^2$ $\therefore a^2 = b^2 + c^2 + 2\ bc$.

But the areas represented by equal numbers are equal, therefore the area denoted by a^2 is equal to the area denoted by $b^2 + c^2 + 2\,bc$.

But this area is equal to the area denoted by b^2 + the area denoted by c^2 + the area denoted by $2\,bc$; that is, to the □ on a line containing b linear units + the □ on a line containing c linear units + twice the rectangle under these two lines. Therefore the square on A B is equal to the sum of the squares on the two parts of A B, together with twice the rectangle under the parts.

PROPOSITION V. (See p. 144.)

Let A C or C B contain a linear units, and C D b linear units. A D contains $a + b$, and D B contains $a - b$ units. $(a + b)\ (a - b) = a^2 - b^2$. Consequently, the space represented by $(a + b)\ (a - b)$ is equal to the space represented by $a^2 - b^2$.

But $(a + b)\ (a - b)$ represents the area of a rectangle whose sides contain respectively $a + b$ and $a - b$ linear units; a^2 represents the □ upon a line containing a linear units, b^2 the □ upon a line containing b linear units.

Therefore the rectangle contained by the sum and the difference of two lines is equal to the difference between the squares on the lines.

PROPOSITION VI. (See p. 147.)

The only difference between this proposition and the last is that b is larger than a, so that we get $(b + a)\ (b - a) = b^2 - a^2$. But in this case, as in the last, $b + a$ is the sum of the lines, and $b - a$ is their difference.

PROPOSITION VII. (See p. 149.)

Let A B contain a linear units, and B C b of the same. A C contains $a - b$ units. $(a - b)^2 = a^2 + b^2 - 2\,ab$ $\therefore$ $(a - b)^2 + 2\,ab = a^2 + b^2$.

Since the numerical expression on the first side of the equation stands for the same number as the numerical expression on the second side, the number of square units (or the quantity of space) represented by the first side is equal to the number of square units (or the quantity of space) represented by the second side.

$(a - b)^2$ represents the □ on A C; $2\,ab$ represents twice the rect. A B, B C; $a^2 + b^2$ represents the sum of the □s on A B and B C. Therefore □ on A B + □ on B C $= 2^{ce}$ rect. A B, B C + □ on A C.

PROPOSITION IX. (See p. 152.)

Let A D contain a, and D B b linear units. A C contains $\frac{a+b}{2}$ and C D contains $\frac{a-b}{2}$ units.

$$\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 = \frac{a^2+b^2+2\,ab}{4} + \frac{a^2+b^2-2\,ab}{4}$$

$$\therefore \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 = \frac{2\,a^2+2\,b^2}{4} = \frac{a^2+b^2}{2}$$

$$\therefore 2\left(\frac{a+b}{2}\right)^2 + 2\left(\frac{a-b}{2}\right)^2 = a^2 + b^2.$$

The quantity of space denoted (numerically) by the first side is equal to the quantity denoted (numerically) by the second side. $a^2 + b^2$ denotes the sum of the squares on lines containing respectively a and b units, that is, the sum of the squares on A D and D B.

$\left(\frac{a+b}{2}\right)^2$ denotes the □ on half the sum of the same lines, and $\left(\frac{a-b}{2}\right)^2$ denotes the □ on half their difference.

Therefore the sum of the □s on A D and D B is equal to twice the □ on A C (half their sum), together with twice the □ on C D (half their difference).

Proposition X. may be proved in a similar manner.

In what precedes it has been assumed that the lines dealt with are *commensurable*, that is, admit of being expressed *exactly* in terms of some common unit. Now it is easy to show that pairs or sets of lines may be found, in endless variety, which have no common measure, so that there are no finite numbers whatsoever that will express them *exactly*. But suppose we have two incommensurable lines, A and B. By taking a linear unit sufficiently small, definite numbers a and b can always be found such that the lengths which they denote differ from A and B by quantities X and Y smaller than any that we choose to name; and the area of the rectangle which contains ab square units differs from that contained by A and B by a quantity less than any that we choose to name. Consequently, although all definite arithmetical calculations respecting such lines and the rectangles formed by them are incapable of absolute exactness, the error involved in treating the calculations as exact may always be made smaller than any quantity that we can name. The full discussion of this question, however, would take us too far for our present purpose.

From the relation between the sides of a rectangle and its area certain obvious inferences may be drawn. Thus if a represent the area of a rectangle and b the length of one side, the length of the other side is $\frac{a.}{b}$

The area of any parallelogram is equal to that of a rectangle having the same base and altitude.

The area of a triangle is half that of a rectangle having the same base and altitude, or to that of a rectangle having the same altitude and a base of half the size.

All rectangles are equal in area when the products of the numbers that represent their sides are equal.

EXERCISES.

[The numbers within the brackets indicate that no proposition is to be used in the Exercise except some one or more of those which are so specified.]

1. A B is a finite right line divided into any two parts at C; D is the middle point of A C; E the middle point of B C. Find the middle point of A B. (I. 1-3.)

2. A B C is an equilateral △. D is a point in B C. Find a point E in A C, and a point F in A B, such, that if these three points be joined, D E F shall be an equilateral triangle. (I. 1-4.)

3. A B is a diameter of a circle, C is its centre, D is a point on the circumference. C and D, A and D, and B and D are joined. Show that in the △ A B D the ∠ at D is equal to the sum of the ∠s at A and B. (I. 1-5.)

4. In the figure on p. 26, let O be the point where A L and F C intersect each other. Prove that A O and C O are equal, and that if B O be drawn, it bisects the ∠ A B C. (I. 1-6.)

5. Construct an isosceles △, having given the base and the length of one of the sides. (I. 1-3.)

6. A B is a finite right line; C is the middle point of it; C D is a line at right angles to A B. Prove that every point* in C D is equidistant from the points A and B. (I. 1-4.)

7. Construct an isosceles △, the base and the sum of the two sides being given. (I. 1-11.)

8. In the △ A B C (p. 26), lines A K and C K are drawn bisecting the ∠s B A C and B C A, and meeting at K. Prove that A K C is an isosceles △. (I. 1-6.)

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* When *every* point in a line satisfies certain conditions, the line is said to be the *locus* of the point that satisfies those conditions. Thus, in this exercise, C D is the *locus* of a point equidistant from A and B.

9. A is a point in a given right line; B is a point outside the line. Find a point in the line equidistant from A and B. (I. 1-11.)

10. A B C is an angle; D is a point situated anywhere you please. Draw through D a line that shall cut off equal parts of A B and B C. (I. 1-11.)

11. Draw the line along which a rectangular leaf must be folded in order that the corner may coincide with a given point on the page. (I. 1-11.)

12. A B is a right line. A line, C D, is drawn from the point C in it in any direction. Show that the lines which bisect the △s A C D and D C B are perpendicular to each other. (I. 1-13.)

13. Find a point which is equidistant from the three vertical points of a △. (I. 1-12.)

14. A ray of light, emitted from a given point, A, is reflected by the mirror B C. Show where the ray must strike the mirror in order to reach another given point, D.* (I. 1-15.)

15. Two isosceles triangles stand on the same base and on the same side of it. Prove that the line which joins their vertices, when produced, will bisect the base and be at right angles to it. (I. 1-8.)

16. The perpendicular is the shortest line that can be drawn from a given point to a given straight line. (I. 1-19.)

17. Of all the lines that can be drawn from a given point to a given straight line, those which make equal angles with the perpendicular are equal to each other, and of any other pair of lines that is the shorter which is the nearer to the perpendicular. (I. 1-19.)

18. Having given the segments of the base of a △ formed by a line which bisects the vertical ∠, and the difference between the other two sides, construct the △. (I. 1-9.)

19. A ray of light emitted from a given point is reflected by a mirror to another given point. Prove that the path which it takes is the shortest that it could take. (I. 1-20.)

20. A point passes from a given position, first to a given

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* The angle of incidence is equal to the angle of reflection.

straight line, and then to a circle on the same side of that line. Determine the shortest path that it can take. (I. 1-20).

21 The two diagonals of any four-sided figure are together less than the sum of the four sides, and greater than half their sum. (I. 1-20.)

22. Construct a △, having given the base, one of the ∠s at the base, and the sum of the other two sides. (I. 1-23.)

23. If a right line be drawn cutting off any part of a polygon, the perimeter of the part that remains is less than that of the original polygon; and if one polygon be described within another, the perimeter of the inner polygon is less than that of the outer. (I. 1-20.)

24. Two equal straight lines bisect each other. Show that the two extremities of either are at the same distance from the other. (I. 1-26.)

25. If two exterior angles of a △ are equal, the △ is isosceles. (I. 1-13.)

26. If two straight lines are parallel to two others, the ∠s which the lines form with each other are either equal or supplementary. (I. 1-29.)

27. Trisect a right angle. (I. 1-32.)

28. Take the figure in Prop. I. (p. 16). Produce A B to meet the right-hand circle in F. Join C and D, C and F, and D and F, and prove that C D F is an equilateral △. (I. 1-32.)

29. If lines be drawn bisecting two ∠s of a △, the point where they meet is equidistant from all the three sides. (I. 1-26.)

30. If right lines be drawn through the vertices of a △ parallel to the opposite sides, a figure will be formed which is divided into four △s equal to each other in all respects. (I. 1-34.)

31. Construct an equilateral △, the sum of the three sides of which shall be equal to a given right line. (I. 1-32.)

32. If two parallel right lines be intersected by a right

line, the lines which bisect the two interior angles on the same side are perpendicular to each other. (I. 1–32.)

33. Describe a rhombus in which two opposite angles shall be double the other two. (I. 1–34.)

34. Find a point in the base of a △, such that the two lines drawn from it parallel respectively to one side and meeting the other, shall be equal. (I. 1–34.)

35. Draw a straight line from one side of a △ to another, so that it shall be parallel and equal to a given straight line. (I. 1-34.)

36. Prove that the diagonals of a parallelogram bisect each other. (I. 1-34.)

37. In a right-angled parallelogram the diagonals are equal. (I. 1-34.)

38. If the opposite sides of a quadrilateral are equal, it is a parallelogram. (I. 1–34.)

39. Describe a parallelogram, the area and perimeter of which shall be equal to those of a given △.

40. Of all △s on the same base and between the same parallels, the isosceles △ is that which has the least perimeter. (I. 1-37.)

41. If from any point in the diagonal of a parallelogram right lines be drawn to the opposite corners, the two △s on one side of the diagonal are respectively equal to the two △s on the other. (I. 1-38.)

42. The line joining the points of bisection of two sides of a △ is parallel to the base. (I. 1-39.)

43. If one side of a △ be divided into any number of equal parts, and lines be drawn from the points of section parallel to the base, and meeting the other side, that side will be divided into the same number of equal parts as the other. (I. 1-34.)

44. Hence, show how to divide a given finite right line into any number of equal parts.

45. Construct a △, having given the base, the sum of the other two sides, and the vertical angle. (I. 1-32.)

46. A point is equally distant from two parallel straight lines. Prove that any line drawn through that point from

one of the parallels to the other is bisected at the point. (I. 1–29.)

47. Show how to draw a straight line through a point within an angle from one side of the angle to the other, so as to be bisected at the point. (I. 1–29.)

48. The lines which join the points of bisection of the three sides of a △ divide it into four △s equal to each other in all respects. (I. 1–39.)

49. If straight lines be drawn through the angles of a quadrilateral figure parallel to its diagonals, they form a parallelogram of which the area is double that of the original figure. (I. 1–41.)

50. If straight lines be drawn through the points of bisection of each pair of conterminous sides of the four sides of any quadrilateral figure, they form a parallelogram, the area of which is half that of the original figure. (I. 1–40.)

51. Construct a △, having given the base, the sum of the other two sides, and a line to which the line bisecting the vertical angle of the △ is parallel. (I. 1–31.)

52. Two mirrors are at right angles to each other. If a ray of light fall on one of them, be reflected to the other, and then reflected from that, the direction of the last part of the ray is parallel to that of the first. (I. 1–28.)

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